Radical Functions

The root of an expression is the reverse of raising it to a power. For example, if you want the square root of an expression, then you want another expression, such that, when you square it, you get what is inside the square root.

Since $12^2 = 144$, the number 12 is the positive square root of 144. This is written $\sqrt{144} = 12$. Also, $\sqrt{400} = 20$, $\sqrt{0.0001} = 0.01$

Definition 1: For positive real numbers a and b, $\sqrt{a} = b \Leftrightarrow a = b^2$ The number b is the positive root

When you answer questions like "What does *N* equal, if $N^3 = 64$?" you are undoing the process of cubing a number. To answer this question, suppose someone has cubed a number and told you the result, and you want to find the original number.

Definition 2: For any real numbers a and b, $\sqrt[3]{a} = b \Leftrightarrow a = b^3$

Every number has one cube root. For cube roots, however, the cube root of a positive number is always positive, and the cube root of a negative number is always negative. So, the symbol $\sqrt[3]{\ }$ represents the one cube root a number has.

$$\sqrt[3]{8} = 2 \qquad \sqrt[3]{-8} = -2$$

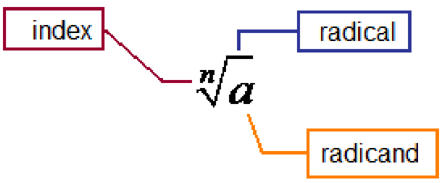
Fourth roots, Fifth roots, sixth roots, and so on are similar to square roots and cube roots.

Definition 3: Definition of the n^{th} root of a number: Let a and b be real numbers and let $n \geq 2$ be a positive integer. If $a = b^n$ then b is the n^{th} root of a. If n=2, the root is a square root. If n=3, the root is a cube root.In general, $\sqrt[n]{\ }$ denotes the n^{th} root of x.

$$\sqrt[n]{a} = b \Leftrightarrow a = b^n$$

Definition 4: Principle n^{th} root of a number: Let a be a real number that has at least one n^{th} root. The principle n^{th} root of a is the n^{th} root that has the same sign as a. It is indicated by a radical symbol $\sqrt[n]{a}$ $\quad$ $\text{Principle } n^{th} \text{ root}$

The positive integer n in the **index** of a radical, and the number a is the radicand.



The square root of x is written $\sqrt{x}$ not $\sqrt[2]{x}$ .If there is no number, the radical is assumed to represent a square root.

Rule 1: When n is even, $\sqrt[n]{x}$ denotes the positive n^{th} root. The negative n^{th} root is $-\sqrt[n]{x}$. When n is even, x must be nonnegative.

$$\cancel{\sqrt{4} = \pm 2} \qquad \text{Correct}: -\sqrt{4} = -2 \text{ and } \sqrt{4} = +2$$

Remark: The n^{th} root of 0 is always 0, no matter what n is.

Example 1: Evaluate each expression:

1) $\sqrt{36}=6$

2) $-\sqrt{36}=-6$

3) $\sqrt[3]{125}=5$

4) $\sqrt[3]{-\frac{8}{125}}=\sqrt[3]{\left(-\frac{2}{5}\right)^3}=-\frac{2}{5}$

5) $\sqrt[4]{625}=\sqrt[4]{(5)^4}=5$

6) $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce -81

The table below shows the generalizations about the n^th roots of a real number($\sqrt[n]{a}$):

Real number a	Integer n	Root(s)	Example
$a>0$	$n>0,\ n\ is\ even$	$\sqrt[n]{a}\,,-\sqrt[n]{a}$	$\sqrt[4]{16}=2\,,-\sqrt[4]{16}=-2$
$a>0\quad or\ \ a<0$	$n\ is\ odd$	$\sqrt[n]{a}$	$\sqrt[3]{-8}=-2$
$a<0$	$n\ is\ even$	No real roots	$\sqrt[4]{-16}$ does not exist
$a=0$	$n\ is\ even\ or\ odd$	$\sqrt[n]{0}=0$	$\sqrt[6]{0}=0$

Integers such as $1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ and\ 100$ are called perfect squares because they have integer square roots. Similarly, integers such as $1,\ 8,\ 27,\ 64, and\ 125$ are called perfect cubes because they have integer cube roots.

Properties of radicals: Let a and b be real numbers, variables or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive numbers

Properties	Examples
1) $\sqrt[n]{a^m}=\left(\sqrt[n]{a}\right)^m=a^{\frac{m}{n}}$	1) $\sqrt[3]{8^2}=\left(\sqrt[3]{8}\right)^2=(2)^2=4$
2) $\sqrt[n]{a}\bullet\sqrt[n]{b}=\sqrt[n]{ab}$	2) $\sqrt[4]{5}\bullet\sqrt[4]{25}=\sqrt[4]{125}$
3) $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}},b\neq 0$	3) $\frac{\sqrt[3]{2}}{\sqrt[3]{5}}=\sqrt[3]{\frac{2}{5}}$
4) $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$	4) $\sqrt[2]{\sqrt[4]{12}}=\sqrt[2\bullet 4]{12}=\sqrt[8]{12}$
5) $\sqrt[n]{a^n}=\left(\sqrt[n]{a}\right)^n=a$	5) $\left(\sqrt[5]{10}\right)^5=10$
6) For n even, $\sqrt[n]{a^n}=\lvert a\rvert$ For n odd, $\sqrt[n]{a^n}=a$	6) $\sqrt{(-12)^2}=\lvert -12\rvert=12$ $\sqrt[5]{(-13)^5}=-13$

Example 2: Use the properties of radicals to simplify each expression:

1) $\sqrt{8}\bullet\sqrt{2}=\sqrt{8\times 2}=\sqrt{16}=4$
2) $\sqrt[6]{6^6}=6$
3) $\sqrt[4]{y^4}=|y|$

Rule 2: Simplifying radicals: An expression involving radicals is in simplest form when the following conditions are satisfied:
1. All possible factors have been removed from under the radical sign.
2. The index on the radical is reduced.
3. All fractions have radical – free denominators (accomplished by a process called rationalizing the denominators)
4. All indicated operations have been performed (if possible)

To simplify a radical, factor the radicand into factors whose exponents are multiples of the indexes. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

Example 3: Simplify

1) $\sqrt[4]{48}$

Perfect 4th power ↓ ; left factor ↓

$$\sqrt[4]{48}=\sqrt[4]{16\bullet 3}=\sqrt[4]{2^4\bullet 3}=\sqrt[4]{2^4}\bullet\sqrt[4]{3}=2\sqrt[4]{3}$$

2) $\sqrt[3]{-40x^6}$

Perfect 4th power ↓ ; left factor ↓

$$\sqrt[3]{-40x^6}=\sqrt[3]{\left(-8x^6\right)\bullet 5}=\sqrt[3]{\left(-2x^2\right)^3\bullet 5}=-2x^2\sqrt[3]{5}$$

Radical Functions

A **radical function** is a function that can be written in the form $f(x)=A(x)$ where $A(x)$ is a radical expression with the variable x in the radicand.
Some examples are: $f(x)=\sqrt{x}$, $g(x)=\sqrt{x-2}$, and $h(x)=\sqrt{x+3}$.

To evaluate a radical function for a particular value, substitute it into the equation and simplify. For even roots, if the substitution produces a negative number under the radical, the function does not have a value for the given input.

Domain of a Radical Function

How do we determine the domain of a radical function?

Case 1: If the radical function has an **even** index, set the radicand greater than or equal to zero, then solve for the independent variable. Since we cannot find the value of a negative radicand with an even index, the values obtained are the domain.

Case 2: If the radical function has an **odd** index, the domain is the set of all real numbers.

Example 4: Find the domain of definition of each radical function

1) $g(x)=\sqrt{x-2}$

The expression under the radical sign is $x-2$

$x-2\geq 0$

$\Rightarrow x\geq 2$

Therefore, the domain of definition is: $x\in[2,+\infty)$

2) $h(x)=\sqrt{x+3}$

The expression under the radical sign is $x+3$

$x+3\geq 0$

$\Rightarrow x\geq -3$

Therefore, the domain of definition is: $x\in[-3,+\infty)$

Example 5: Arrange the following expressions in order from smallest to largest:

$$64^{\frac{2}{3}},\ 64^{\frac{1}{2}},\ 64^{-\frac{1}{3}},\ \left(\frac{1}{64}\right)^{-\frac{3}{2}}$$

Evaluate each expression then compare the obtained values

- $64^{\frac{2}{3}} = \left(\sqrt[3]{64}\right)^2 = (4)^2 = 16$
- $64^{\frac{1}{2}} = \sqrt{64} = 8$
- $64^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$
- $\left(\frac{1}{64}\right)^{-\frac{3}{2}} = \frac{1^{-\frac{3}{2}}}{64^{-\frac{3}{2}}} = \frac{1}{64^{-\frac{3}{2}}} = 64^{\frac{3}{2}} = \left(\sqrt{64}\right)^3 = (8)^3 = 512$

Write in order from smallest to largest: $64^{-\frac{1}{3}},\ 64^{\frac{1}{2}},\ 64^{\frac{2}{3}},\ \left(\frac{1}{64}\right)^{-\frac{3}{2}}$

Example 6: Simplify:

$$\sqrt[3]{16x^5}$$

- 16 can be factored to 2×8 where 8 is a perfect cube.
- x^5 can be factored to $x^3 \times x^2$ where x^3 is a perfect cube.

$\sqrt[3]{16x^5}$
$= \sqrt[3]{2 \times 8 \times x^3 \times x^2}$
$= \sqrt[3]{2} \times \sqrt[3]{8} \times \sqrt[3]{x^3} \times \sqrt[3]{x^2}$
$= \sqrt[3]{2} \times 2 \times x \times \sqrt[3]{x^2}$
$= 2x\sqrt[3]{2x^2}$

Example 7: Simplify

$$\sqrt[3]{-54x^8 y^7 z^9}\ .$$

- -54 can be factored to 2×-27 where -27 is a perfect cube.
- x^8 can be factored to $x^6 \times x^2$ where x^6 is a perfect cube.
- y^7 can be factored to $y^6 \times y$ where y^6 is a perfect cube.
- z^9 is a perfect cube.

$\sqrt[3]{-54x^8 y^7 z^9}$
$= \sqrt[3]{2 \times -27 \times x^6 \times x^2 \times y^6 \times y \times z^9}$
$= \sqrt[3]{2} \cdot \sqrt[3]{-27} \cdot \sqrt[3]{x^6} \cdot \sqrt[3]{x^2} \cdot \sqrt[3]{y^6} \cdot \sqrt[3]{y} \cdot \sqrt[3]{z^9}$
$= \sqrt[3]{2} \times (-3) \times x^2 \times \sqrt[3]{x^2} \times y^2 \times \sqrt[3]{y} \times z^3$
$= -3x^2 y^2 z^3 \sqrt[3]{2x^2 y}$

Example 8: Simplify

$$\sqrt[3]{-64x^7}$$

$\sqrt[3]{-64x^7}$
$= \sqrt[3]{(-4)^3 x^6 \cdot x}$
$= -4x^3 \sqrt[3]{x}$