Quadratic Equations

Quadratic equations are used in many areas of science and engineering. The path of a projectile (e.g. a cannon ball) is (almost) parabolic, and we use a quadratic equation to find out where the projectile is going to hit. Also, parabolic antennas are another application.

Quadratic Equation: The general form of a quadratic equation is $ax^2 + bx + c = 0$ where *x* is the variable and *a*, *b* & *c* are constants and $a \neq 0$

I. Factoring

Solving a quadratic (or any kind of equation) by factoring depends on the use of a principle known as the zero-product rule.

Rule 1: **Zero Product Rule:** If $ab = 0$ then either $a = 0$ or $b = 0$ (or both)
Thus, if you can factor an expression that is equal to zero, then you can set each factor equal to zero and solve it for the unknown.

Example 1: Solve: $x^2 - x = 6$

Given:	$x^2 - x = 6$
Move all terms to one side:	$x^2 - x - 6 = 0$
Factor:	$(x - 3)(x + 2) = 0$
Set each factor equal to zero and solve:	$(x - 3) = 0$ **OR** $(x + 2) = 0$
Solutions:	$x = 3$ **AND** $x = -2$

No Constant Term i.e. c=0.
If a quadratic equation has no constant term (i.e. $c = 0$) then it can easily be solved by factoring out the common *x* from the remaining two terms:

$$ax^2 + bx = 0$$

$$x(ax + b) = 0$$

Then, using the zero-product rule, you set each factor equal to zero and solve to get the two solutions: $x = 0 \quad or \quad x = -b/a$

Remark 1: Do not divide out the common factor of *x* or you will lose the $x = 0$ solution. Keep all the factors and use the zero-product rule to get the solutions.

Trinomials: When a quadratic has all three terms, you can still solve it with the zero-product rule if you are able to factor the trinomial. A more thorough discussion of factoring trinomials may be found in the chapter on polynomials

II. Solving by Square Roots: No First-Degree Term i.e. b=0

If the quadratic has no linear, or first-degree term (i.e. b = 0), then it can be solved by isolating the x^2 and taking square roots of both sides:

$$ax^2 + c = 0$$

$$ax^2 = -c$$

$$x^2 = -\frac{c}{a}$$

$$x = \pm\sqrt{-\frac{c}{a}}$$

Rule 2: The square root property:

The solution of $ax^2 = k$ is $\left\{-\sqrt{\frac{k}{a}}, \sqrt{\frac{k}{a}}\right\}$

III. Completing the Square

The technique of completing the square is to take a trinomial that is not a perfect square, and make it one by inserting the correct constant term (which is the square of half the coefficient of x). Of course, inserting a new constant term has to be done in an algebraically legal manner, which means that the same thing needs to be done to both sides of the equation. This is best demonstrated with an example.

Example 2: Solve $x^2 + 6x - 2 = 0$

Given Equation:	$x^2 + 6x - 2 = 0$
Move original constant to other side:	$x^2 + 6x = 2$
Add new constant to both sides (the square of half the coefficient of x):	$x^2 + 6x + 9 = 2 + 9$
Write left side as perfect square:	$(x+3)^2 = 11$
Square root both sides (remember to use plus-or-minus):	$x + 3 = \pm\sqrt{11}$
Solve for x:	$x = -3 \pm \sqrt{11}$

Remark 2: If the Coefficient of x^2 is Not 1: First divide through by the coefficient, then proceed with completing the square.

Example 3: Solve $2x^2 + 3x - 2 = 0$

Given Equation:	$2x^2+3x-2=0$
Divide through by coefficient of x^2: (in this case a 2)	$\frac{1}{2}\left(2x^2+3x-2=0\right)$ $x^2+\frac{3}{2}x-1=0$
Move constant to other side:	$x^2+\frac{3}{2}x=1$
Add new constant term: (the square of half the coefficient of x, in this case 9/16):	$x^2+\frac{3}{2}x+\frac{9}{16}=1+\frac{9}{16}$
Write as a binomial squared: (the constant in the binomial is half the coefficient of x)	$\left(x+\frac{3}{4}\right)^2=\frac{25}{16}$
Square root both sides: (remember to use plus-or-minus)	$x+\frac{3}{4}=\pm\frac{5}{4}$
Solve for x:	$x=\frac{-3\pm5}{4}$

Thus $\quad x=-2 \text{ and } x=\frac{1}{2}$

Example 4: Solve

1) $x^2+6x-16=0$.

$$x^2+6x-16=0$$
$$x^2+2\cdot3\cdot x=16$$
$$x^2+2\cdot3\cdot x+3^2=16+3^2$$
$$(x+3)^2=25$$
$$x+3=\pm5$$
$$x_1=2 \text{ and } x_2=-8.$$

2) $2x^2+4x-9=0$.

$$2x^2+4x-9=0$$
$$x^2+2x=\frac{9}{2}$$
$$x^2+2x+1=\frac{9}{2}+1$$
$$(x+1)^2=\frac{11}{2}$$
$$x+1=\pm\frac{\sqrt{22}}{2}$$
$$x=-1\pm\frac{\sqrt{22}}{2}.$$

IV. The Quadratic Formula

Deriving the Quadratic Formula
The quadratic formula can be derived by using the technique of completing the square on the general quadratic formula:

Given:	$ax^2+bx+c=0$
Divide through by *a*:	$x^2+\frac{b}{a}x+\frac{c}{a}=0$
Move the constant term to the right side:	$x^2+\frac{b}{a}x=-\frac{c}{a}$
Add the square of one-half the coefficient of *x* to both sides:	$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$
Factor the left side (which is now a perfect square), and rearrange the right side:	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$
Get the right side over a common denominator:	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Take the square root of both sides (remembering to use plus-or-minus):	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Solve for *x*:	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Rule 3: For $ax^2+bx+c=0$ where $a\neq 0$, the value of x is given by:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The formula requires you to take the square root of the expression $b^2 - 4ac$, which is called the *discriminant* because it determines the nature of the solutions.

$$\Delta=b^2-4ac$$

<u>Discussing discriminant</u>
There are three cases for $\Delta=b^2-4ac$:

1) If Δ is positive $\Rightarrow$ the expression under the square root is positive and therefore there are two distinct solutions.

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

2) If Δ is zero $\Rightarrow$ the expression under the square root sign is zero and therefore the value of the square root is zero too. This means that regardless of whether you add that zero or subtract it, you get the same result. So there is only one root.

$$x = -\frac{b}{2a}$$

3) If Δ is less than zero: as you know, there is no real number whose square is negative. Therefore, in such cases there are no real solutions.

$\Delta > 0$	two distinct solutions
$\Delta = 0$	only one root
$\Delta < 0$	no real solutions

<u>To solve any equation using the quadratic formula, follow the steps listed below:</u>

Step 1: Make sure that the equation is written in form ax^2 +bx+c = 0.

Step 2: Make sure that the right part is zero

Step 3: Write down the values of a, b and c. Make sure that you get the signs right.

Step 4: Calculate the discriminant, $\Delta = b^2 - 4ac$. Again, do not forget about signs

Step 5: If the discriminant is negative, stop. There is no real solution.

Step 6: If the discriminant is not negative, calculate the solution according to the formula:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Example 5: Solve for x: $2x^2 + 5x - 5 = 0$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(2)(-5)}}{2(2)}$$

$$\Rightarrow x = \frac{-5 \pm \sqrt{65}}{4}$$

$$\Rightarrow x_1 = \frac{-5 + \sqrt{65}}{4} \quad and \quad x_2 = \frac{-5 - \sqrt{65}}{4}$$

Example 6: Solve: $x^2 - x - 12 = 0$.

We put *a* = 1, *b* = –1 and *c* = –12 to the quadratic formula. Then we have

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-12)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{49}}{2}$$
$$x_1 = 4 \text{ and } x_2 = -3.$$

Example 7: Solve: $x^2 - 2x + 2 = 0$.

We put *a* = 1, *b* = –2 and *c* = 2 to the quadratic formula. Then we have

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{-4}}{2}$$
$$= \frac{2 \pm 2\sqrt{-1}}{2}$$
$$= 1 \pm \sqrt{-1}.$$

There are no real roots