Probability

Let a random experiment have sample space S. Any assignment of probabilities to events must satisfy some basic laws of probability:

1) For any event A, P(A) ≥ 0.
2) P(S) = 1. Where S is the sample space
3) $0 \le P(A) \le 1$

Definition 1: The **theoretical probability** of an event is defined as the ratio:

$$\frac{\text{number of favorable cases}}{\text{total number of possible cases}}$$

$$P(A) = \frac{n(A)}{n(S)}$$

Definition 2: **Sample Space**: The collection of all possible outcomes

Definition 3: **Simple Event**: One specific outcome in the sample space.

Example 1: Rolling a die. Rolling a 2 is a simple event (a possible outcome). The sample space would be rolling a 1, 2, 3, 4, 5, or 6.

Notes:

- Since $\phi \subset S$ and S $\subset$ S, both ϕ = { } and S are events.
- ϕ = { } is called the impossible event and has probability zero, i.e., P(ϕ) = 0.
- S is called the definite event and has probability of 1, i.e., P(S) = 1.
- Probability of any other event, say A, is between zero and one, i.e., 0 ≤ P(A) ≤ 1 for any event A.

Set notation and set algebra, such as $\cup, \cap, \in$, and complement ($A' = A^c = \bar{A}$) are used in defining some events.

To achieve an understanding of the laws of probability, it helps to have a concrete example in mind.

Activity 1: Consider a single roll of two dice, a red one and a green one. The table below shows the set of outcomes in the sample space, S. Each outcome is a pair of numbers--the number appearing on the red die and the number appearing on the green die. The event that consists of the whole sample space is the event that some one of the outcomes occurs. This event is certain to happen; if we roll the dice, the outcome cannot be something other than one of the 36 outcomes listed in the table. Therefore, the probability associated with the event S is P(S) = 1.

Number on Green Die						
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
	1	2	3	4	5	6

Number on Red Die

If the dice are fair, then each of the 36 possible outcomes is equally likely. Also only one of the outcomes can happen on any roll of the dice. Consequently, to find the probability of the event consisting of just one of the outcomes (any one), we simply divide the probability of the entire sample space by 36.
Therefore, for example:

P(1,1) = $\frac{1}{36}$

P(4,5) = $\frac{1}{36}$

P(2,3) = $\frac{1}{36}$

Any event is a subset of the entire sample space. SO, the sample space of the given event is less than or equal to the entire sample space. For each event, the probability will be calculated in two different ways.

Example 2: let A be the event that the sum of the numbers on the dice is 7. This event consists of 6 of the possible outcomes:

A = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}.

The event A that the sum of the numbers on the dice is 7 is the same as the event that the outcome of the roll is (1,6) **or** (2,5) **or** (3,4) **or** (4,3) **or** (5,2) **or** (6,1). The probability that event A happens is:

Method 1:

$n(A) = 6$

$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

Method 2:

P(A) = P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1)

$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}$

$= \frac{6}{36} = \frac{1}{6}$

Example 3: let the event B be the event that the number showing on the green die is 1. This event may also be described as the event that the outcome of the roll of the dice is (1,1) **or** (1,2) **or** (1,3) **or** (1,4) **or** (1,5) **or** (1,6), as . The probability that event B happens is:

Method 1:

$n(B)=6$

$P(B)=\frac{n(B)}{n(S)}=\frac{6}{36}=\frac{1}{6}$

Method 2:

P(B) = P(1,1) + P(1,2) + P(1,3) + P(1,4) + P(1,5) + P(1,6)

$= \frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}$

$= \frac{6}{36}=\frac{1}{6}$

Example 4: Let the event C be the event that the number showing on the green die is 6. This event may also be described as the event that the outcome of the roll of the dice is (6,1) **or** (6,2) **or** (6,3) **or** (6,4) **or** (6,5) **or** (6,6). The probability that event C happens is:

Method 1:

$n(C)=6$

$P(C)=\frac{n(C)}{n(S)}=\frac{6}{36}=\frac{1}{6}$

Method 2:

P(C) = P(6,1) + P(6,2) + P(6,3) + P(6,4) + P(6,5) + P(6,6)

$= \frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}$

$= \frac{6}{36}=\frac{1}{6}$

Example 5: Let the event D be the event that neither one of the numbers appearing on the dice is greater than 4. This event may also be described as the event that the outcome of the roll of the dice is (1,1) **or** (1,2) **or** (1,3) **or** (1,4) **or** (2,1) **or** (2,2) **or** (2,3) **or** (2,4) **or** (3,1) **or** (3,2) **or** (3,3) **or** (3,4) **or** (4,1) **or** (4,2) **or** (4,3) **or** (4,4). The probability of this event is:

Method 1:

$n(D)=16$

$P(D)=\frac{n(D)}{n(S)}=\frac{16}{36}=\frac{4}{9}$

Method 2:

P(D) = P(1,1) + P(1,2) + P(1,3) + P(1,4) + P(2,1) + P(2,2) + P(2,3) + P(2,4) + P(3,1) + P(3,2) + P(3,3) + P(3,4) + P(4,1) + P(4,2) + P(4,3) + P(4,4)

= 16/36

= 4/9.

Activity 2: **Rolling two dice to find the sum**

The sample space for the total of each roll is {2,3,4,5,6,7,8,9,10,11,12}.

The above listing is NOT equally likely, since most of the sums can occur in more than one way.

For example, a total of 8 occur from rolling 6 & 2, 5 & 3, 4 & 4, 3 & 5, 2 & 6 while 12 occurs only when you roll 6 & 6.

An equally likely sample space for this event is given in this table.

All Possible Outcomes for Rolling Two Dice					
1&1	2&1	3&1	4&1	5&1	6&1
1&2	2&2	3&2	4&2	5&2	6&2
1&3	2&3	3&3	4&3	5&3	6&3
1&4	2&4	3&4	4&4	5&4	6&4
1&5	2&5	3&5	4&5	5&5	6&5
1&6	2&6	3&6	4&6	5&6	6&6

Corresponding Sums for Rolling Two Dice					
2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

The above chart helps us to calculate probabilities for this experiment:
P(rolling a 2) = 1/36 = 1/36
P(rolling a 3) = 2/36 = 1/18
P(rolling a 4) = 3/36 = 1/12
P(rolling a 5) = 4/36 = 1/9
P(rolling a 6) = 5/36 = 5/36
P(rolling a 7) = 6/36 = 1/6
P(rolling an 8) = 5/36 = 5/36
P(rolling a 9) = 4/36 = 1/9
P(rolling a 10) = 3/36 = 1/12
P(rolling an 11) = 2/36 = 1/18
P(rolling a 12) = 1/36 = 1/36
P(rolling an even sum) = 18/36 = 1/2

Activity 3: Rolling two dice to find the product

The sample space for the total of each roll is
{1,2,3,4,5,6,8,9,10,12,15,16,18,20,24,25,30,36}.

The above listing is NOT equally likely, since most of the products can occur in more than one way. For example, a product of 12 occurs from rolling 2 & 6, 3 & 4, 4 & 3, 6 & 2 while 9 occurs only when you roll 3 & 3.

The tables below list the sample space so that it is equally likely.

All Possible Outcomes for Rolling Two Dice					
1&1	2&1	3&1	4&1	5&1	6&1
1&2	2&2	3&2	4&2	5&2	6&2
1&3	2&3	3&3	4&3	5&3	6&3
1&4	2&4	3&4	4&4	5&4	6&4
1&5	2&5	3&5	4&5	5&5	6&5
1&6	2&6	3&6	4&6	5&6	6&6

Corresponding Products for Rolling Two Dice					
1	2	3	4	5	6
2	4	6	8	10	12
3	6	9	12	15	18
4	8	12	16	20	24
5	10	15	20	25	30
6	12	18	24	30	36

Based on the results presented in the above tables, we can find the probabilities:
P(Rolling an even product) = 27/36 = .75
P(Rolling an odd product) = 9/36 = .25
P(Rolling a 6) = 4/36 = .111
P(Rolling an 18) = 2/36 = .055