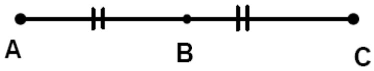
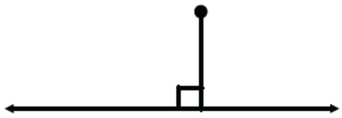
Perpendicular and Bisectors of a Triangle

In this section we will see some words that we need to define, those words are of important use to locate a point and to understand the relationships between lines and points

Definition 1: **Equidistant**: a point is equidistant from two points if its distance from each point is the same.



Definition 2: **Distance from a point to a line**: the length of the perpendicular segment from the point to the line.



Definition 3: **Equidistant from the two lines**: when a point is the same distance from one line as it is from another line.

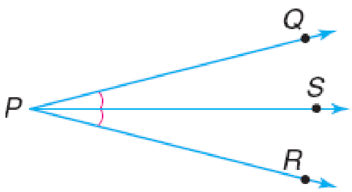
Part A: Bisectors of the Angles of a Triangle

Recall that the bisector of an angle is a ray that separates the angle into two congruent angles.

$\overrightarrow{PS}$ bisects $\angle QPR$

$\angle QPS \cong \angle SPR$

$m\angle QPS = m\angle SPR$

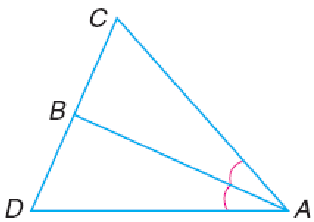


Definition 4: An angle bisector of a triangle is a segment that divides an angle of the triangle into two congruent angles. One of the endpoints of an angle bisector is a vertex of the triangle, and the other endpoint is on the side opposite that vertex.

$\overline{AB}$ is an angle bisector of □DAC

$\angle DAB \cong \angle CAB$

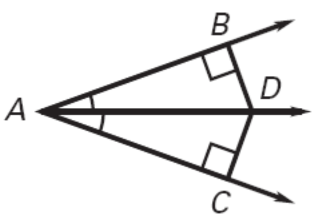
$m\angle DAB = m\angle CAB$



Theorem 1: **Angle Bisector Theorem**: if a point is on the bisector of an angle, then it is equidistant from the two sides of the angle.

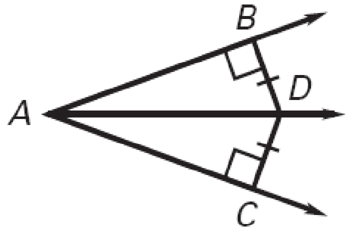
$m\angle BAD = m\angle CAD$

$\Rightarrow BD = DC$

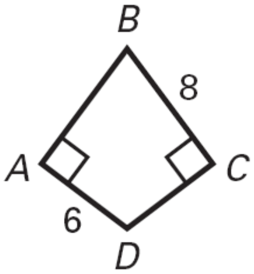


Converse: **Converse of the Angle Bisector theorem**: if a point is in the interior of an angle and is equidistant from the sides of the angle, then it lies on the bisector of the angle.

$BD = DC$
$\Rightarrow m\angle BAD = m\angle CAD$



Example 1: In the diagram, *D* is on the bisector of $\angle ABC$. What is the measure of DC? Explain.
D is on the bisector of $\angle ABC$
$\Rightarrow DA = DC$
$\Rightarrow DC = 6$



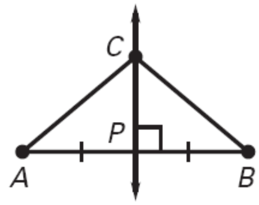
Part B: Perpendicular Bisectors of the Sides of a Triangle

Definition 5: **Perpendicular Bisectors**: a segment, ray, line or plane that is perpendicular to a segment at its midpoint.

Theorem 2: **Perpendicular Bisector Theorem**: if a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

CP is the perpendicular bisector of line segment $\overline{AB}$

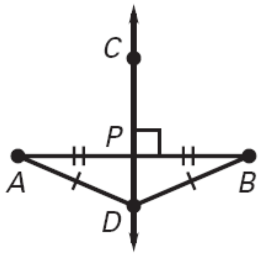
$\Rightarrow CA = CB$



Converse: **Converse of the Perpendicular Bisector Theorem**: if a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

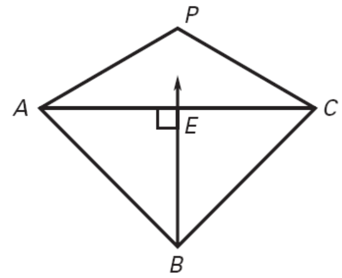
$DA = DB$

$\Rightarrow$ point D lies on the perpendicular bisector of line segment $\overline{AB}$



Example 2:

Given: $\overrightarrow{BE} \perp$ bisector of $\overline{AC}$



Prove: $\triangle ABE \cong \triangle CBE$

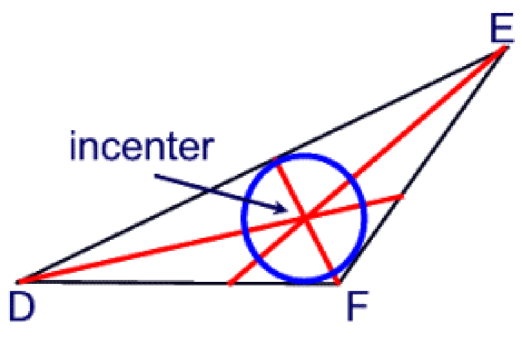
Proof:

Statements	Reasons
In $\triangle ABE$ and $\triangle CBE$, we have:	
$\overrightarrow{BE} \perp$ bisector of $\overline{AC}$	Given
$\overline{BA} \cong \overline{BC}$	**Perpendicular Bisector Theorem**
$m\angle AEB = m\angle CEB = 90^\circ$	Definition of $\perp$ lines
$\overline{BE} \cong \overline{BE}$	Reflexive property
$\triangle ABE \cong \triangle CBE$	HL theorem

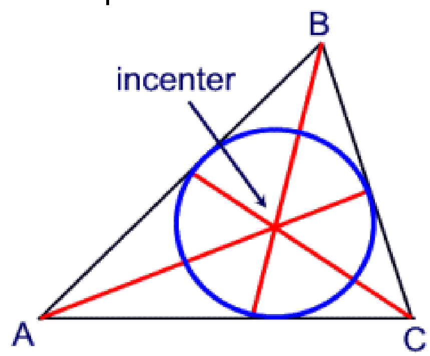
Two arbitrary lines will intersect at a point: unless the lines happen to be parallel, which is unusual. Thus concurrency is an expected property of two lines.
But it is rare that three lines have a point in common. One of the aspects of advanced Euclidean geometry is the fact that so many triples of lines determined by triangles are concurrent. Each of the triangle centers in this section and the coming sections is an example of that phenomenon.

Definition 6: CONCURRENT LINES: Three lines are *concurrent* if there is a point P such that P lies on all the three lines. The point P is called the *point of concurrency*. Three segments are concurrent if they have an interior point in common.

Definition 7: The angle bisectors of a triangle are concurrent. The point of concurrence is the center of an inscribed circle within the triangle. The point of concurrence is called the incenter.



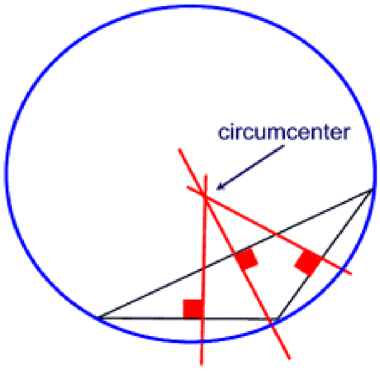
Angle Bisectors of an obtuse triangle.



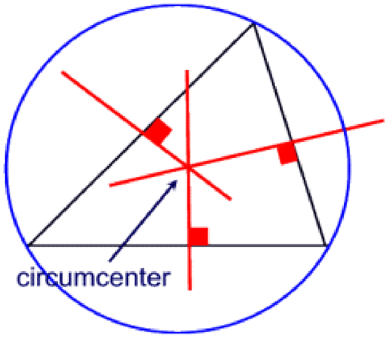
Angle Bisectors of an acute triangle.

Notice that the point of concurrence is in the interior of the triangles.

Definition 8: The perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrence is the center of a circumscribed circle about the triangle. The point of concurrence is called the circumcenter.



Perpendicular Bisectors of an obtuse triangle.

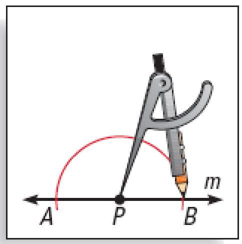
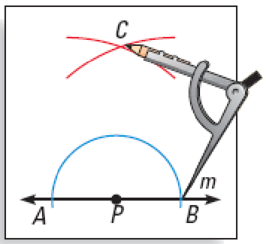


Perpendicular Bisectors of an acute triangle.

<u>Notice that the point of concurrence is not necessarily in the interior of the triangles.</u>

Construction: Perpendicular Through a point on a line

Use these steps to construct a line that is perpendicular to a given line m and passing through a given point P on m.

Place the compass point at P. Draw an arc that intersects line m twice. Label the intersections as A and B.	
Use a compass setting greater than AP. Draw an arc from A. With the same setting, draw an arc from B. Label the intersection of the arcs as C.	
Use a straightedge to draw $\overleftrightarrow{CP}$. This line is perpendicular to line m and passes through P.