Operations with Rational Functions

Part A: Multiplication and Division

When we multiply or divide two rational expressions, the algebra that is used is simply an application of normal fraction manipulation.

The rules used to multiply and divide rational expressions are the same as those used to multiply and divide rational numbers.

Procedure To Multiply Rational Expressions
Step 1: Find the domain of definition
Step 2: Factor the numerators and denominators.
Step 3: Cancel all pairs of factors common to the numerators and denominators.
Step 4: Multiply the numerators, and then multiply the denominators.

$$\frac{A}{B}\times\frac{C}{D}=\frac{A\bullet C}{B\bullet D} \qquad where \quad B\neq 0 \ and\ D\neq 0$$

You can also do the cancelling *before or after* you multiply and still get the same answer.

Example 1: Find the product of:

$$f(x)=\frac{x(x-4)}{x-3} \text{ and } g(x)=\frac{x(x-3)}{(2x+1)(x-4)}$$

$$f(x)\times g(x)=\frac{x(x-4)}{x-3}\times\frac{x(x-3)}{(2x+1)(x-4)} \qquad where\, x\neq 3\,; x\neq 4\, and\ x\neq -\frac{1}{2}$$

Method 1: $f(x)\times g(x)=\frac{x\cancel{(x-4)}}{\cancel{(x-3)}}\times\frac{x\cancel{(x-3)}}{(2x+1)\cancel{(x-4)}}=\frac{x^2}{2x+1}$

Method 2: $f(x)\times g(x)=\frac{x(x-4)}{x-3}\times\frac{x(x-3)}{(2x+1)(x-4)}=\frac{x^2\cancel{(x-4)}\cancel{(x-3)}}{\cancel{(x-3)}(2x+1)\cancel{(x-4)}}=\frac{x^2}{(2x+1)}$

Procedure: To Divide Rational Expressions
Step 1: Find the domain of definition
Step 2: Invert the second rational expression and change $\div$ to $\times$.
Step 3: Multiply.

$$\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\times\frac{D}{C}=\frac{A\bullet D}{B\bullet C} \qquad where \quad B\neq 0\,; C\neq 0\, and\ D\neq 0$$

Division of rational functions follows this general idea. It is oftentimes summarized using the mnemonic “STAY-CHANGE-FLIP.”

Example 2: Find $f(x) \div g(x)$ given that:

$$f(x)=\frac{x-6}{2x+2} \text{ and } g(x)=\frac{(2x-12)}{(x+1)(3x+5)}$$

$$f(x) \div g(x)$$

$$=\frac{x-6}{2x+2} \div \frac{(2x-12)}{(x+1)(3x+5)} \qquad \textit{where } x \neq -1 \textit{ and } x \neq -\frac{5}{3}$$

$$=\frac{x-6}{2x+2} \times \frac{(x+1)(3x+5)}{(2x-12)} \qquad x \neq 6$$

$$=\frac{\cancel{x-6}}{2\cancel{(x+1)}} \times \frac{\cancel{(x+1)}(3x+5)}{2\cancel{(x-6)}}$$

$$=\frac{3x+5}{4}$$

Part B: Addition and Subtraction

To add or subtract two rational expressions we have to make sure that the terms have the same denominator.

Two cases will be discussed:

1) Rational expressions having the same denominators
2) Rational expressions having different denominators

Case 1: If the denominators are the same, just add or subtract the numerators.
Step 1: Find the domain of definition
Step 2: Combine the numerators together.
Step 3: Put the sum or difference found in step 1 over the common denominator.
Step 4: Reduce to lowest terms if possible

$$\frac{A}{B}+\frac{C}{B}=\frac{A+C}{B} \qquad B \neq 0$$

Example 3: Add: $\frac{2x+9}{x-3}+\frac{x-5}{x-3}$

Here the denominators are the same, so we just add the numerators.

$$\frac{2x+9}{x-3}+\frac{x-5}{x-3} \qquad x \neq 3$$

$$=\frac{(2x+9)+(x-5)}{x-3}$$

$$=\frac{2x+9+x-5}{x-3}$$

$$=\frac{3x+4}{x-3}$$

<u>Case 2:</u> If the denominators are not the same, we have to find a proper LCM. And then we adjust the numerators.

Step 1: Find the domain of definition

Step 2: Factor all the denominators

Step 3: The LCD is the list of all the DIFFERENT factors in the denominators raised to the highest power that there is of each factor.

Step 4: Write equivalent fractions using the LCD if needed.
If we multiply the numerator and denominator by the exact same expression it is the same as multiplying it by the number 1. If that is the case, we will have equivalent expressions when we do this.
Now the question is WHAT do we multiply top and bottom by to get what we want? We need to have the LCD, so you look to see what factor(s) are missing from the original denominator that is in the LCD. If there are any missing factors then that is what you need to multiply the numerator AND denominator by.

Step 5: Combine the rational expressions

Step 6: Reduce to lowest terms if possible

$$\frac{A}{B}+\frac{C}{D}=\frac{AD+CB}{BD} \qquad B\neq 0 \,\&\, D\neq 0$$

Example 4: Add: $\frac{3x-1}{x+1}+\frac{2x-1}{x+3}$

Here the denominators are not the same, so we need to find the LCD, then find a common denominator.

$$\frac{3x-1}{x+1}+\frac{2x-1}{x+3} \qquad x\neq -3, x\neq -1$$

$$=\frac{(3x-1)(x+3)+(2x-1)(x+1)}{(x+3)(x+1)}$$

$$=\frac{3x^2+9x-x-3+2x^2+2x-x-1}{(x+3)(x+1)}$$

$$=\frac{5x^2+11x-4}{(x+3)(x+1)}$$

If there are no powers, we just include all of the factors that are there, without any repetition, so we get the product of the three factors for our LCD.