Multiplying Matrices

We can only multiply matrices if the number of columns in the first matrix is the same as the number of rows in the second matrix.
Multiplying a 2 × 3 matrix by a 3 × 4 matrix is possible and it gives a 2 × 4 matrix as an answer.
Multiplying a 7 × 1 matrix by a 1 × 2 matrix is okay; it gives a 7 × 2 matrix
A 4 × 3 matrix times a 2 × 3 matrix is NOT possible.

How to Multiply 2 Matrices?

We use letters first to see what is going on. We'll see an example with numbers after.
As an example, let's take a general 2 × 3 matrix multiplied by a 3 × 2 matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

The answer will be a 2 × 2 matrix.

Step 1: We multiply and add the elements as follows. We work **across** the 1st row of the first matrix, multiplying **down** the 1st column of the second matrix, element by element. We **add** the resulting products. Our answer goes in position a_{11} (top left) of the answer matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} au + bw + cy & \\ & \end{bmatrix}$$

Step 2: We do a similar process for the 1st row of the first matrix and the **2nd** column of the second matrix. The result is placed in position a_{12}.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} au + bw + cy & av+bx+cz \\ & \end{bmatrix}$$

Step 3: Now for the **2nd** row of the first matrix and the **1st** column of the second matrix. The result is placed in position a_{21}.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} au + bw + cy & av+bx+cz \\ du+ew+fy & \end{bmatrix}$$

Step 4: Do the 2nd row of the first matrix and the 2nd column of the second matrix. The result is placed in position a_{22}.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} au+bw+cy & av+bx+cz \\ du+ew+fy & dv+ex+fz \end{bmatrix}$$

So the result of multiplying our 2 matrices is as follows:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} au+bw+cy & av+bx+cz \\ du+ew+fy & dv+ex+fz \end{bmatrix}$$

The process is the same for any size matrix. We multiply **across** rows of the first matrix and **down** columns of the second matrix, element by element. We then add the products:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

In this case, we multiply a 2 × 2 matrix by a 2 × 2 matrix and we get a 2 × 2 matrix as the result.

Remark: Be careful with **writing** matrix multiplication. The following expressions have *different meanings:*

1) *AB* is **matrix multiplication**
2) *A*×*B* **cross** product, which returns a **vector**
3) *A*•*B* **dot** product, which returns a **scalar**

Example 1:

If $A = \begin{pmatrix} 0 & -1 & 2 \\ 4 & 11 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 6 & 1 \end{pmatrix}$, find *AB* and *BA* if possible:

Now *AB* is (2 × 3)(3 × 2) which will give 2 × 2:

$$AB = \begin{pmatrix} (0)(3)+(-1)(1)+(2)(6) & (0)(-1)+(-1)(2)+(2)(1) \\ (4)(3)+(11)(1)+(2)(6) & (4)(-1)+(11)(2)+(2)(1) \end{pmatrix}$$

$$AB = \begin{pmatrix} 11 & 0 \\ 35 & 20 \end{pmatrix}$$

Now *BA* is (3 × 2)(2 × 3) which will give 3 × 3:

$$BA = \begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 4 & 11 & 2 \end{pmatrix} = \begin{pmatrix} 3(0)-1(4) & 3(-1)-1(11) & 3(2)-1(2) \\ 1(0)+2(4) & 1(-1)+2(11) & 1(2)+2(2) \\ 6(0)+1(4) & 6(-1)+1(11) & 6(2)+1(2) \end{pmatrix}$$

$$= \begin{pmatrix} 0-4 & -3-11 & 3-2 \\ 0+8 & -1+22 & 2+4 \\ 0+4 & -6+11 & 12+2 \end{pmatrix} = \begin{pmatrix} -4 & -14 & 1 \\ 8 & 21 & 6 \\ 4 & 5 & 14 \end{pmatrix}$$

$$BA = \begin{pmatrix} -4 & -14 & 1 \\ 8 & 21 & 6 \\ 4 & 5 & 14 \end{pmatrix}$$

So in this case, $AB \neq BA$

Example 2:
Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 4 & 6 \end{bmatrix}$$

Find (a) *AB* (b) *BA*

(a) $AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1\cdot 2+2\cdot 3 & 1\cdot 3+2\cdot 4 & 1\cdot 0+2\cdot 6 \\ -1\cdot 2+3\cdot 3 & -1\cdot 3+3\cdot 4 & -1\cdot 0+3\cdot 6 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 12 \\ 7 & 9 & 18 \end{bmatrix}$

(b) *BA* is not defined.