Laws of Exponents

Repeated multiplication can be written in exponential form
In general, if a is a real number, variable, or algebraic expression and n is a positive integer, then:

$$a^n = \underbrace{a \bullet a \bullet a \bullet a \bullet a \ldots a}_{n\ factors}$$

where n is the exponent and a is the base. The expression a^n is read "a to the n^{th} power." An exponent can be negative as well.

Properties for exponents:

Property 1: $a^m \times a^n = a^{m+n}$
Example 1: $a^2 \times a^3 = (a \times a) \times (a \times a \times a) = a^5$

Property 2: $a^m \div a^n = a^{m-n}$
Example 2: $a^7 \div a^4 = \dfrac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a} = a^3$

Property 3: $(a^m)^n = a^{m \bullet n}$
Example 3: $(a^3)^4 = (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a) = a^{12}$

Remark: It is important to recognize the difference between expressions such as $(-2)^4$ and -2^4. In $(-2)^4$, the parenthesis indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. So, $(-2)^4 = 16$, whereas $-2^4 = -16$.

<u>Some special cases</u>

It can be sown that the three rules above are true for any values of m and n, provided that $a \neq 0$, but it is not possible for us to prove this yet. However, by using powers which are whole numbers we can see how some particular cases will have to go.

1- $a^5 \div a^4 = \dfrac{a \times a \times a \times a \times a}{a \times a \times a \times a} = a^1 = a$ by rule 2: $a^5 \div a^4 = a^{5-4} = a$
So we must have: $a^1 = a$

2- $a^4 \div a^4 = \dfrac{a \times a \times a \times a}{a \times a \times a \times a} = 1$ by rule 2: $a^4 \div a^4 = a^{4-4} = a^0$
So, we must have: $a^0 = 1$

3- $a^3 \div a^4 = \frac{a \times a \times a}{a \times a \times a \times a} = \frac{1}{a}$ by rule 2: $a^3 \div a^4 = a^{3-4} = a^{-1}$

So, we must have: $a^{-1} = \frac{1}{a}$

In fact, more generally, $a^{-n} = \frac{1}{a^n}$

4- $a^{1/2} \times a^{1/2} = a^1$ by rule 1: $a^1 = a$

So $a^{1/2}$ is the number which multiplied by itself gives a

$a^{1/2}$ means the square root of a

Similarly, $a^{1/3} \times a^{1/3} \times a^{1/3} = a^1$ by rule (1)

So, $a^{1/3}$ means the cube root of a, or $\sqrt[3]{a}$ and $a^{1/n}$ means the n^{th} root of a, or $\sqrt[n]{a}$

Example 1: Write the following expressions without negative exponents and then evaluate

1) $7^{-3} = \frac{1}{7^3} = \frac{1}{343}$

2) $\left(\frac{3}{2}\right)^{-2} = \frac{3^{-2}}{2^{-2}} = \frac{2^2}{3^2} = \frac{4}{9}$ or $\frac{1}{\left(\frac{3}{2}\right)^2} = \frac{1}{\frac{3^2}{2^2}} = \frac{1}{\frac{9}{4}} = \frac{4}{9}$

Example 2: Simplify

1) $\left(\frac{1}{16}\right)^{\frac{3}{4}} \div \left(\frac{1}{16}\right)^{\frac{1}{4}}$.

$\left(\frac{1}{16}\right)^{\frac{3}{4}} \div \left(\frac{1}{16}\right)^{\frac{1}{4}}$

$= \left(\frac{1}{16}\right)^{\frac{3}{4}-\frac{1}{4}}$ *Apply the quotient rule*

$= \left(\frac{1}{16}\right)^{\frac{1}{2}}$ *Simplify*

$= \sqrt{\frac{1}{16}} = \frac{1}{4}$ *Express in radical form*

2) $-\left(9^{\frac{5}{4}}\right)^{\frac{6}{5}}$.

$-\left(9^{\frac{5}{4}}\right)^{\frac{6}{5}}$

$= -(9)^{\frac{5}{4}\times\frac{6}{5}}$ — *Apply the product of a product rule*

$= -9^{\frac{6}{4}}$ — *Simplify*

$= -9^{\frac{3}{2}}$ — *Simplify*

$= -\left(\sqrt{9}\right)^{3}$ — *Write in radical form*

$= -3^{3} = -27$ — *Simplify*

3) $\dfrac{\left(\sqrt[5]{\dfrac{1}{-32}}\right)^{3}}{\sqrt[5]{\dfrac{1}{-32}}}$.

$\dfrac{\left(\sqrt[5]{\dfrac{1}{-32}}\right)^{3}}{\sqrt[5]{\dfrac{1}{-32}}}$

$= \dfrac{\left(\dfrac{1}{-32}\right)^{\frac{3}{5}}}{\left(\dfrac{1}{-32}\right)^{\frac{1}{5}}}$ — *Change to exponential form*

$= \left(\dfrac{1}{-32}\right)^{\frac{3}{5}-\frac{1}{5}}$ — *Apply the quotient rule*

$= \left(\dfrac{1}{-32}\right)^{\frac{2}{5}}$ — *Simplify*

$= \left(\sqrt[5]{\dfrac{1}{-32}}\right)^{2}$ — *Write in radical form*

$= \left(\dfrac{1}{-2}\right)^{2} = \dfrac{1}{4}$ — *Simplify*

4) $\left(x^{\frac{3}{4}} \cdot y^{\frac{2}{3}}\right)^{\frac{6}{7}}$

$\left(x^{\frac{3}{4}} \cdot y^{\frac{2}{3}}\right)^{\frac{6}{7}}$

$= x^{\frac{3}{4}\times\frac{6}{7}} \cdot y^{\frac{2}{3}\times\frac{6}{7}}$ *Apply the power of a product rule*

$= x^{\frac{9}{14}} \cdot y^{\frac{4}{7}}$ *Simplify*

$= x^{\frac{9}{14}} y^{\frac{4}{7}}$ *Simplify*

Example 3: Simplify the expression and evaluate if $x = 256$.

$\left(\dfrac{x^{\frac{3}{4}}}{x^{\frac{1}{4}}}\right)^{\frac{1}{2}}$

$= \left(\dfrac{x^{\frac{3}{8}}}{x^{\frac{1}{8}}}\right)$ *Apply the power of a quotient rule*

$= x^{\frac{1}{4}}$ *Apply the quotient rule*

$= (256)^{\frac{1}{4}}$ *Substitute* $x = 256$

$= \sqrt[4]{256} = 4$ *Write in radical form*