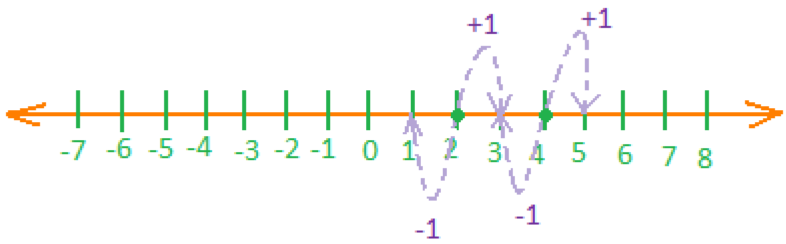
# Inequalities

A statement that has a quantity is either greater than or less than but not equal to the other. Instead of using an equal sign (=) as in an equation, these symbols are used: > (greater than) and < (less than) or ≥ (greater than or equal) and ≤(less than or equal to).

**Axioms and properties:** For all real numbers a, b, and c
**Axiom 1:** one of the relations a < b, a = b, a>b is true.
**Axiom 2:** Transitivity: If a > b, and b > c, then a > c. Also, if a < b and b < c then a < c.

**Fundamental properties:**
1) If a > b, then a + c > b + c
This can be seen geometrically, since c shifts both a and b the same distance (to the right when c is positive and to the left when c is negative). Thus their relative positions are the same before and after shifting.
For example: 4 > 2 so 4 + 1 > 2 + 1 and 4 - 1 > 2 -1




2) If a > b and c > d then a + c > b + d

3) i) If a> b and c > 0 then $a \bullet c > b \bullet c$
ii) If a > b and c < 0 then $a \bullet c < b \bullet c$

**Note 1:** What (3 ii) shows is that when an inequality is multiplied (or divided) by a negative number, the inequality is reversed.

4) If a and b are either both positive or both negative and a > b, then $\frac{1}{a} < \frac{1}{b}$

**<u>Solving inequalities:</u>** Rules to follow when solving inequalities

The first two rules are the same rules we use to solve equations:

**Rule 1:** We may add or subtract any quantity from both sides of the inequality and keep the same set of solutions to the inequality.

**Rule 2:** We may multiply or divide both sides of the inequality by any positive number and keep the same set of solutions.

The third rule is unique to inequalities

**Rule 3:** If we multiply or divide both sides of the inequality by a negative number, we need to reverse the direction of the inequality to keep the same solutions.

**Writing Solutions to Inequalities**

The solution to an inequality is a set of numbers. For example, $x > 5$ describes the set of all numbers that are strictly greater than 5. This set would include numbers like 6, 237, 5.01, and $\frac{11}{2}$. When the solution to an inequality is written, there are two standard ways of writing the solution.

**Set-Builder Notation**

To write $x > 5$ formally in set notation, we would write $\{x \mid x > 5\}$. This is read as "the set of *x*-values such that *x* is greater than 5." The curly brackets denote that we're writing a set of numbers. The vertical line (|) is read as "such that."

**Interval Notation**

To write $x > 5$ in interval notation, we would write $(5, \infty)$. The problem with this notation is that you have to be careful not to mix it with a coordinate point. With this notation, writing (a,b) means {x | $a < x < b$}. Notice that with $x > 5$, there was no upper value, so infinity ($\infty$) was used to denote that the value of x could be as big as we want. If the inequality is not a strict inequality, then we use square brackets. For example $2 \le x < 4$ would be written [2,4). Notice the square bracket is used for the inequality that includes the "or equal to," and the round bracket is used for the strict inequality.

**Example 1:** Solve $-7x + 4 < 18$, $x \in \{-2, 0, 1, 3\}$. Graph the solution set.

$-7x + 4 < 18$

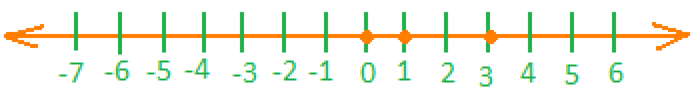
$-7x < 18 - 4$ i. e. $-7x < 14$

Dividing throughout by $-7$, we get

$\frac{-7x}{-7} > \frac{14}{-7}$

$x > -2$

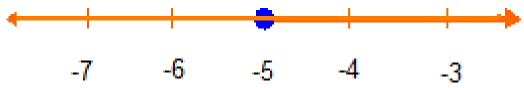
$\Rightarrow$Solution set = {0, 1, 3}




**Note 2:**

1) When graphing inequalities involving, only integer dots are used.
2) When graphing inequalities involving real numbers, lines, rays and dots are used.
3) A dot is used if the number is included. A hollow dot is used if the number is not included.

Example 2: Solve 2 x + 10 < 3 (x +5) , $x \in$ □ . Graph the solution set.
2 x + 10 < 3 x + 15
2 x - 3 x < 15 - 10
- x < 5
x > -5 (on multiplying by -1)




**Compound Inequalities**

A compound inequality like $2 < x < 4$ describes a set of *x*-values between two numbers. You can think of this as saying $2 < x$ *and* $x < 4$. The only values for *x* that satisfy this compound inequality are numbers that satisfy both parts of the inequality.

When you solve compound inequalities, you can either carefully work with all three parts at once, or you can break the compound inequality into two pieces, work with each separately, then rewrite your answer as a compound inequality in the end.

Example 3: Solve $7 \le 4x + 2 \le 12$, $x \in$ □ .

$7 \le 4x + 2 \le 12$ Subtracting - 2 throughout

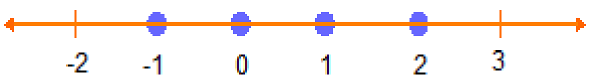
$7 - 2 \le 4x + 2 - 2 \le 12 - 2$

$5 \le 4x \le 10$ Dividing by 4 throughout

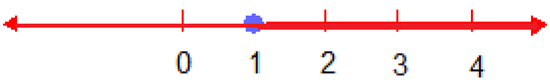
$\frac{5}{4} \le x \le \frac{10}{4}$

$1.25 \le x \le 2.5$

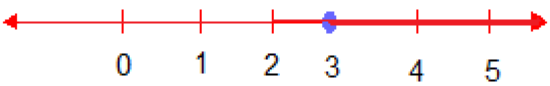
Example 4: Graph the set of x such that -1 ≤ x ≤2 and x is an integer
{ x |-1 ≤x≤ 2, x is an integer }



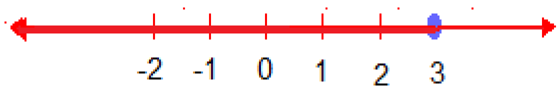

Example 5: Graph:
1) x such that x ≥1: { x | x ≥1 }



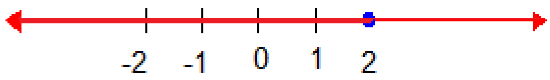

2) x such that x > 3: { x | x > 3 }



3) Graph the set of x such that x < 3
{ x | x < 3}




4) Graph the set of x such that x ≤ 2
{x | x ≤ 2}




**Example 6:** Solve $2 \le \frac{3x-5}{4} < 5$

First we rewrite this statement into two inequalities and then solve

$$2 \le \frac{3x-5}{4} < 5$$

$$\begin{array}{lcl}
2 \le \frac{3x-5}{4} & \text{and} & \frac{3x-5}{4} < 5 \\
8 \le 3x-5 & \text{and} & 3x-5 < 20 \\
8+5 \le 3x-5+5 & \text{and} & 3x-5+5 < 20+5 \\
13 \le 3x & \text{and} & 3x < 25 \\
\frac{13}{3} \le x & \text{and} & x < \frac{25}{3}
\end{array}$$

$$\Rightarrow \frac{13}{3} \le x < \frac{25}{3}$$

The solution is $x \in \left[\frac{13}{3}, \frac{25}{3}\right)$

Example 7: Solve $2x+1\leq 4x-4<x-7$.

$$2x+1\leq 4x-4<x-7$$
$$2x+1\leq 4x-4 \quad \textit{and} \quad 4x-4<x-7$$
$$2x+1-4x-1\leq 4x-4-4x-1 \quad \textit{and} \quad 4x-4-x+4<x-7-x+4$$
$$-2x\leq -5 \quad \textit{and} \quad 3x<-3$$
$$x\geq \frac{-5}{-2} \quad \textit{and} \quad x<\frac{-3}{3}$$
$$x\geq \frac{5}{2} \quad \textit{and} \quad x<-1$$

The solution is $x\in\left[\frac{5}{2},+\infty\right)$ and $x\in(-\infty,-1]$ impossible

Example 8: If $4\leq x\leq 8$ and $3\leq y\leq 5$ find the minimum valueof $\frac{2y}{3+x}$

$\frac{2y}{3+x}$ This expression would have minimum value when numerator is minimum and the denominator is maximum.
Minimum value of y is 3
Maximum value of x is 8

Therefore, minimum value of $\frac{2y}{3+x}=\frac{2(3)}{3+8}=\frac{6}{11}$

Example 9: Solve $2<3x-1<6$

**Method 1:**

$2<3x-1<6$ Add 1 to all parts of the inequality
$3<3x<7$ Divide all parts by 3
$1<x<\frac{7}{3}$

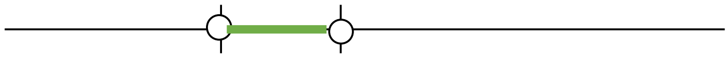
**Method 2:** Start by breaking the inequality into two pieces:

| | | |
|---|---|---|
| $2 < 3x - 1$ | $3x - 1 < 6$ | Add 1 to both sides of each inequality |
| $3 < 3x$ | $3x < 7$ | Divide both sides of each by 3 |
| $1 < x$ | $x < \frac{7}{3}$ | |

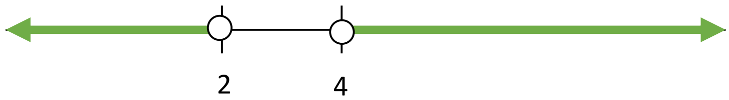
Either way, we put the answer together in the end as:

Set notation: $\{x \mid 1 < x < \frac{7}{3}\}$

Interval notation: $(1, \frac{7}{3})$



**Important note:** Suppose the shaded area on the number line below represents the set of values you're trying to describe with an inequality




Writing $4 < x < 2$ is **not correct notation**. First of all, it would suggest that 4 < 2, which is clearly not true. More importantly, $4 < x < 2$ would mean "the set of *x* values for which 4 is less than *x* **and** *x* is less than 2." It is not possibly for *x* to be **both**. Here, the correct way to write this inequality is: $\{x \mid x < 2 \quad or \quad x > 4\}$.