Determinants

We define the **determinant** of a 2×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ as the curiously cross product $a_{11}a_{22} - a_{12}a_{21}$ (We multiply the diagonals (top left × bottom right first), then subtract.). Our symbols for the determinant are $\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

The symbol looks like an absolute value symbol, but in the context of a discussion it is $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, there should not be any confusion. Also, notice that is an array of numbers, while $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is definitely a number. Do not inter mix them! We have special vocabulary related to the value of the determinant.

1) If $\det(A) = |A| = 0$, then A is called a singular matrix
2) If $\det(A) = |A| \neq 0$, then A is non – singular matrix

Rule 1: The **determinant** of a 2×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is:

$$\det(A) = |A| = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 1: Find the determinant of

1) $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

$$\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 4\times3 - 2\times1 = 12 - 2 = 10$$

The final result is a single **number**.

2) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Don't make it hard! The result is $|A|$ =1(4) −2(3) =−2 .

3) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

The result is $|A|$ =1(4) −2(2)=0 .

3 × 3 Determinants

A 3 × 3 determinant is $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Each element, or number, in this determinant is associated with 2×2 determinant called its minor. The **minor** of an element is the determinant that remains when you cross out the row and column containing that element.

The minor of element a_2 is:

$$\begin{vmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ b_1 & \cancel{b_2} & b_3 \\ c_1 & \cancel{c_2} & c_3 \end{vmatrix} \rightarrow \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$

You can evaluate a 3 × 3determinant by expanding by the minors of the elements in any row or column. Consider the position of each element in the original determinant and find its corresponding position on the checkerboard pattern:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

This pattern is used in adding and subtracting the product of the row(column) elements with their corresponding minors.

Example 2: Evaluate:

$$\begin{vmatrix} 5 & 2 & 8 \\ 3 & 4 & 1 \\ 7 & -1 & 6 \end{vmatrix}$$

Expand by minors of the first row.

$$\begin{vmatrix} 5 & 2 & 8 \\ 3 & 4 & 1 \\ 7 & -1 & 6 \end{vmatrix} = 5\begin{vmatrix} 4 & 1 \\ -1 & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 7 & 6 \end{vmatrix} + 8\begin{vmatrix} 3 & 4 \\ 7 & -1 \end{vmatrix}$$

$$= 5(25) - 2(11) + 8(-31)$$

$$= -145$$

The evaluation of determinants can be made easier by transforming the determinant into an equivalent one in which a row (column) has several zero elements. To do this, multiply any row(column)by a constant and add the resulting products to the elements of another row (column).

$$\begin{vmatrix} 2 & 5 & 3 & 7 \\ 4 & 1 & 0 & 2 \\ 3 & 2 & 2 & 1 \\ -2 & 8 & 1 & 0 \end{vmatrix} \rightarrow \begin{cases} Add\ (\text{-}3 \times \text{row 4}) \text{ to row 1} \\ \text{Add } (\text{-}2 \times \text{row 4}) \text{ to row 3} \end{cases} \rightarrow \begin{vmatrix} 8 & -19 & 0 & 7 \\ 4 & 1 & 0 & 2 \\ 7 & -14 & 0 & 1 \\ -2 & 8 & 1 & 0 \end{vmatrix}$$

Once you have a row(column) with several zeros, you can use a 4×4 checkboard pattern of $(+)$ and $(-)$ signs. Since column 3 in the determinant at the right above now contains three zeros, we will expand by its minors:

$$\begin{vmatrix} 8 & -19 & 0 & 7 \\ 4 & 1 & 0 & 2 \\ 7 & -14 & 0 & 1 \\ -2 & 8 & 1 & 0 \end{vmatrix}$$

$$= 0\begin{vmatrix} 4 & 1 & 2 \\ 7 & -14 & 1 \\ -2 & 8 & 0 \end{vmatrix} - 0\begin{vmatrix} 8 & -19 & 7 \\ 7 & -14 & 1 \\ -2 & 8 & 0 \end{vmatrix} + 0\begin{vmatrix} 8 & -19 & 7 \\ 4 & 1 & 2 \\ -2 & 8 & 0 \end{vmatrix} - 1\begin{vmatrix} 8 & -19 & 7 \\ 4 & 1 & 2 \\ 7 & -14 & 1 \end{vmatrix}$$

$$= -1\begin{vmatrix} 8 & -19 & 7 \\ 4 & 1 & 2 \\ 7 & -14 & 1 \end{vmatrix}$$

Do not confuse a **matrix** with a **determinant** which uses vertical bars | |. A **matrix** is a *pattern of numbers*; a determinant gives us a *single number*.