Define and Represent a Function

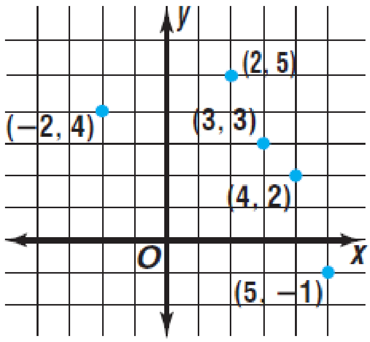
A set of ordered pairs is a **relation**. The set of all first coordinates of the ordered pairs, or *x*-coordinates, is called the **input**. The set of all second coordinates, or *y*-coordinates, is called the **output**

Example 1: Express the relation {(-2, 4), (2, 5), (3, 3), (4, 2), (5, -1)} as a table and as a graph.

As a table, we will list all the x-values and y-values :

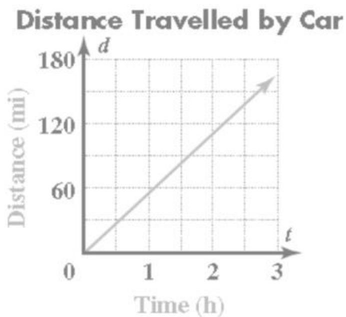
x	*y*
−2	4
2	5
3	3
4	2
5	−1

As a graph, plot the points that represent the given ordered pairs:



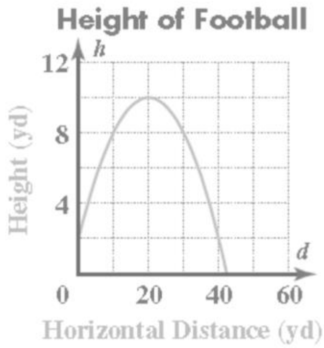
Many of the relationships you have studied have a special name: they are called *functions*. In mathematics, a **function** is a relationship between an input variable and an output variable in which there is only one output for each input.

Activity 1: A car traveling along a highway at 55 miles per hour for *t* hours will cover a distance of 55*t* miles. This can be represented by the equation *d* = 55*t*.



In the car example, the input variable is the time spent on the highway. The output variable is the distance traveled. Since there can be only one distance traveled for any given time, the relationship is a function. In this case, the distance traveled is a *function of* the time.

Activity 2: When a quarterback throws a football, the height of the ball in yards when it has traveled d yards might be described by the equation $h = 2 + 0.8d - 0.02d^2$.



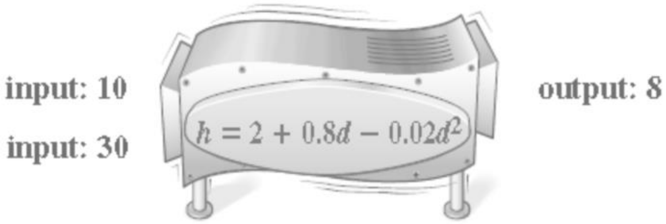
In the football example, the input variable is the horizontal distance the ball has traveled, and the output variable is the ball's height. Since there can be only one height for any given horizontal distance, the relationship is a function. In this case, the height is a *function of* the horizontal distance.

Definition 1: A *function* f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain (or set of inputs) of the function f, and the set B contains the range (or set of outputs)

One way to think about a function is to imagine a machine that takes some input a number, a word, or something else (depending on what the function is) and produces an output.

For example, suppose you put 10 into a function machine for the football example. Since the machine is a function, the output must be *unique*. If you put 10 into the machine, it can give an output of 8, but it can't give both 8 and some other number.

For a function machine, the output must be consistent. That is, the machine will always give the same output for the same input. If you get an output of 8 for an input of 10, then every time you put 10 into the machine, the output will be 8.
It *is* possible that two (or more) inputs will produce the same output. For example, the football-height function machine will produce 8 when you put 10 or 30 into it.



Remark: If more than one output is possible for a given input, the relationship is *not* a function. For example, a machine that outputs the square roots of a positive number can't be a function, because every positive number has *two* square roots.

Example 2: Determine if the given relation {(5, 10), (10, 10), (15, 10)} is function or not.

We need to ask ourselves, does every first element (or input) correspond with **EXACTLY ONE** second element (or output)? In this case, the answer is yes. 5 only goes with 10, 10 only goes with 10, and 15 only goes with 10.

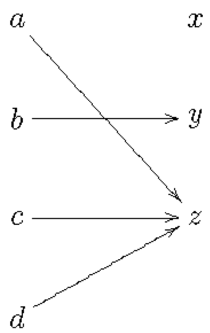
Note that a relation can still be a function if an output value associates with more than one input value as shown in this example. But again, it would be a no no the other way around, where an input value corresponds to two or more output values.

So, this relation would be an example of a function.

Characteristics of a Function from Set A to Set B

1. Each element of A must be matched with an element of B.
2. Some elements of B may not be matched with any element of A.
3. Two or more elements of A may be matched with the same element of B.
4. An element of A (the domain) cannot be matched with two different elements of B.

Example 3: Let A = {a, b, c, d} and B = {x, y, z}. The function f is defined by the relation pictured below:



Example 4: Decide whether the relation represents y as a function of x.

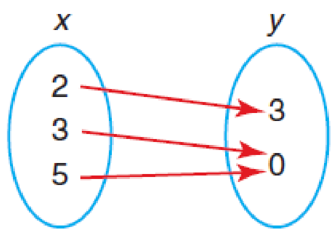
Input (x)	2	2	3	4	5
Output (y)	11	13	8	5	1

This table does not describe y as a function of x because the input 2 is matched with two different y – values.

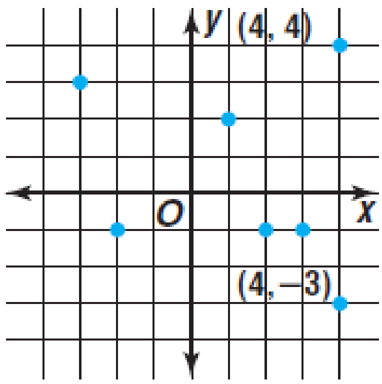
It is common to represent functions by equations or formulas involving variables. For instance, the equation $y = x^2$ represents the variable y as a function of the variable x. In this equation x is the independent variable and y is the dependent variable.

Determine whether each relation is a function. Explain your answer.

{(5, 2), (3, 5), (-2, 3), (-5, 1)}
Since each element of the domain is paired with *exactly* one element of the range, this relation is a function.



The mapping between *x* and *y* represents a function since there is *only one* corresponding element in the range for each element of the domain. *It does not matter if two elements of the domain are paired with the same element in the range.*



The graph represents a relation that is not a function. Look at the points with the ordered pairs (4, -3) and (4, 4). The member 4 in the domain is paired with both -3 and 4 in the range.

Example 5: Which of the equations represent y as a function of x?

1) $y + x^2 = 1$

To determine whether y is a function of x, try to solve for y in terms of x.
Solving for y yields to: $y + x^2 = 1 \Rightarrow y = -x^2 + 1$
Each value of y corresponds to exactly one value of y. So, y is a function of x.

2) $-x+y^2=1$

To determine whether y is a function of x, try to solve for y in terms of x.
Solving for y yields to:

$$-x+y^2=1$$

$$\Rightarrow y^2=x+1$$

$$\Rightarrow y=\pm\sqrt{x+1}$$

The ± indicates that for a given value of x there correspond two values of y. If x=3 then y=±2. So, y is not a function of x.

Function Notation: Traditionally, functions are referred to by the letter *f*, but *f* need not be the only letter used in function names. The following are but a few of the notations that may be used to name a function: *f* (*x*), *g*(*x*), *h*(*a*), *A*(*t*), ...

$f(x)=x^2$ $\quad g(x)=2x+4$ $\quad h(a)=a^2+5a+4$ $\quad A(t)=16t^2-4t-1$

Note: The *f* (*x*) notation can be thought of as another way of representing the *y*-value, especially when graphing. The *y*-axis may even be labeled as the *f* (*x*) axis.

Evaluating Function: To evaluate a function, simply replace the function's variable (substitute) with the indicated number or expression.

Example 6: A function is represented by *f* (*x*) = 2*x* + 5. Find *f* (3).

To find *f* (3), replace the *x*-value with 3.

f (3) = 2(3) + 5 = 11.

The answer, 11, is called the image of 3 under *f* (*x*).

Example 7: Find the value of $h(x)=2x^2+6x-3$ when $x=4a$

Replace the *x*-values with 4*a*.

$$h(4a)=2(4a)^2+6(4a)-3$$

$$\Rightarrow h(4a)=32a^2+24a-3$$

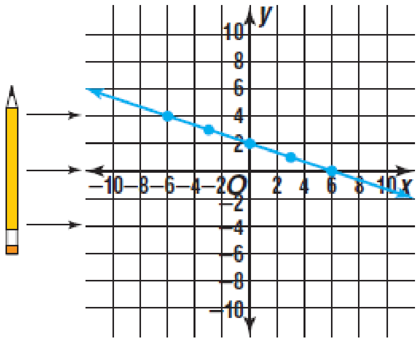
Notice that the final answer is in terms of *a*.

Example 8: Find $f(3h+2)$ when $f(x)=x^2+2x-1$

To find *f* (3*h*+2), replace the *x*-values with 3*h* + 2. Using parentheses for this substitution will help prevent algebraic errors. Use (3*h* + 2) when substituting.

$$f\left(3h+2\right)=\left(3h+2\right)^2+2\left(3h+2\right)-1$$
$$\Rightarrow f\left(3h+2\right)=9h^2+12h+4+6h+4-1$$
$$\Rightarrow f\left(3h+2\right)=9h^2+18h+7$$

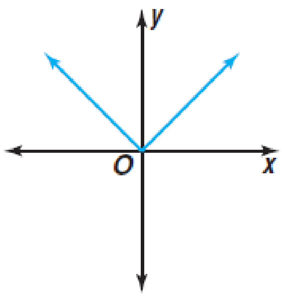
To determine whether an equation is a function, you can use the **vertical line test** on the graph of the equation. Consider the graph below. To perform the test, place a pencil at the left of the graph to represent a vertical line. Move it to the right across the graph.
For each value of *x*, this vertical line passes through exactly one point on the graph. So, the equation is a function.



Rule: If any vertical line passes through no more than one point of the graph of a relation, then the relation is a function.

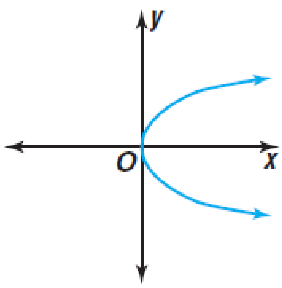
Example 9: Use the vertical line test to determine whether each relation is a function.

A.



The relation is a function since any vertical line passes through no more than one point of the graph of the relation.

B.



The relation is not a function since a vertical line can pass through more than one point.