Binomial Expansion

Powers of polynomials and binomial expansion

Products of such polynomials are easy to calculate and the identities are helpful but what about finding the product of $(x+11)^{14}$. What to do??? Do you really have to multiply this expression times itself 14 times?? That could take forever.
Note the following patterns:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

What **observations** can we make in general about the expansion of $(a+b)^n$?

1) The expansion is a series (an adding of terms).
2) The number of terms in each expansion is one more than n.(terms= $n+1$)
3) The power of *a* starts with a^n and decreases by one in each successive term ending with a^0. The power of *b* starts with b^0 and increases by one in each successive term ending with b^n.
4) The power of *b* is always one less than the "number" of the term. The power of *a* is always n minus the power of *b*.
5) The sum of the exponents in each term adds up to n.
6) The coefficients of the first and last terms are each one.
7) The coefficients of the middle terms form an interesting (but perhaps not easily recognized) pattern where each coefficient can be determined from the previous term. The coefficient is the product of the previous term's coefficient and a's index, divided by the number of that previous term.

Check it out: $a^4+4a^3b+6a^2b^2+4ab^3+b^4$

The second term's coefficient is determined by a^4: $\frac{(1)(4)}{1}=4$

The third term's coefficient is determined by $4a^3b$: $\frac{(4)(3)}{2}=6$

(This pattern will eventually be expressed as a combination of the form C_k^n.)

To Get Coefficient: From the Previous Term: $\frac{(coefficient)(index\ of\ a)}{number\ of\ terms}$

8) Another famous pattern is also developed regarding the coefficients. If the coefficients are "pulled off" of the terms and arranged, they form a triangle known as Pascal's triangle. (The use of Pascal's triangle to determine coefficients can become tedious when the expansion is to a large power.)

Pascal's Triangle: Rows start and end with 1, and each of the other numbers is equal to the sum of the two numbers above it.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

By pulling these observations together with some mathematical rules, a theorem is formed relating to the expansion of binomial terms:

Theorem 1: **Binomial Theorem**: (or Binomial Expansion Theorem)

In the expansion of $(a+b)^n$, for any positive integer *n*: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Definition 1: The $\binom{n}{k}$ notation is just another way of writing a combination such as ${}_nC_k$ (read "*n* choose *k*"). $\binom{n}{k} = {}_nC_k = \frac{n!}{(n-k)!k!}$

Our pattern to obtain the coefficient using the previous term (in observation #6), $\frac{(coefficient)(index\ of\ a)}{number\ of\ terms}$, actually leads to the ${}_nC_k$ used in the binomial theorem.

Definition 2: *n*! represents the product of the first *n* positive integers i.e. $n! = n(n-1)(n-2)...(3)(2)(1)$. We say *n*! as '***n factorial***'

Here is the connection. Using our coefficient pattern in a general setting, we get:

$$(a+b)^n = 1 \cdot a^n + \frac{n}{1} a^{n-1} \cdot b + \frac{n(n-1)}{1 \cdot 2} a^{n-2} \cdot b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3} \cdot b^3 + ... + b^n$$

Theorem 2: The Binomial Theorem can also be written in its expanded form as:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

Remember that $\binom{n}{k} = {}_nC_k = \dfrac{n!}{(n-k)!k!}$ and that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$

Example 1: Expand:

1) $(x+7)^6$

Comparing $(x+7)^6$ with $(a+b)^n$ we conclude that a=x, b=7, n=6

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \Rightarrow (x+7)^6 = \sum_{k=0}^{6}\binom{6}{k}x^{6-k}7^k$$

$$(x+7)^6 = \binom{6}{0}x^6 7^0 + \binom{6}{1}x^{6-1}7^1 + \binom{6}{2}x^{6-2}7^2 + \binom{6}{3}x^{6-3}7^3 + \ldots + \binom{6}{6-1}x^1 7^{6-1} + \binom{6}{6}x^0 7^6$$

$$(x+7)^6$$

$$= x^6 + (6)(7)x^5 + (15)(49)x^4 + (20)(343)x^3 + (15)(2401)x^2 + (6)(16807)x^1 + (1)(117649)x^0$$

$$= x^6 + 42x^5 + 735x^4 + 6{,}860x^3 + 36{,}015x^2 + 100{,}842x^1 + 117{,}649$$

2) $\left(x^4 - 2y^3\right)^5$

Comparing $\left(x^4 - 2y^3\right)^5$ with $(a+b)^n$ we conclude that a= x^4, b= $-2y^3$, n=5

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \Rightarrow \left(x^4-2y^3\right)^5 = \sum_{k=0}^{5}\binom{5}{k}\left(x^4\right)^{5-k}\left(-2y^3\right)^k$$

Note that here it is a MUST that you use parenthesis for the first and second terms

$$\left(x^4-2y^3\right)^5$$

$$= \binom{5}{0}\left(x^4\right)^5\left(-2y^3\right)^0 + \binom{5}{1}\left(x^4\right)^{5-1}\left(-2y^3\right)^1 + \binom{5}{2}\left(x^4\right)^{5-2}\left(-2y^3\right)^2 + \binom{5}{3}\left(x^4\right)^{5-3}\left(-2y^3\right)^3 + \binom{5}{5-1}\left(x^4\right)^1\left(-2y^3\right)^{5-1} + \binom{5}{5}\left(x^4\right)^0\left(-2y^3\right)^5$$

$$= x^{20} + 5x^{16}\left(-2y^3\right) + 10x^{12}\left(4y^6\right) + 10x^8\left(-8y^9\right) + (5)x^4\left(16y^{12}\right) + \left(-32y^{15}\right)$$

$$= x^{20} - 10x^{16}y^3 + 40x^{12}y^6 - 80x^8y^9 + 80x^4y^{12} - 32y^{15}$$

What if I need to **find just "one" term** in a binomial expansion, such as just the 5^{th} term of $\left(6x^2-3\right)^{14}$?

Let's call the term we are looking for the ***r*** th term. From our observations, we know that the coefficient of this term will be ${}_nC_{r-1}$, the power of *b* will be *r* - 1 and the power of *a* will be *n* minus the power of *b*. Putting this information together gives us a formula for the ***r*** th term:

The *r* th term of the expansion of $(a+b)^n$ is: $\binom{n}{r-1}a^{n-(r-1)}b^{r-1}$

Some important points to note about the r^{th} term:

1) The top number of the binomial coefficient is *n*, which is the exponent on your binomial.
2) The bottom number of the binomial coefficient is *r* - 1, where *r* is the term number.
3) *a* is the first term of the binomial and its exponent is *n* - *r* + 1, where n is the exponent on the binomial and *r* is the term number.
4) *b* is the second term of the binomial and its exponent is *r* - 1, where *r* is the term number.

Example 2: What is the coefficient of x^3 in the expansion of $(x+3)^8$?

We are looking for the coefficient of x^3 which is the 6th term $\Rightarrow r=6$

So, the 6th term is $\binom{8}{6-1}x^{8-(6-1)}(3)^{6-1}=\binom{8}{5}x^3(3)^5$

Therefore, the coefficient of x^3 is $\binom{8}{5}(3)^5=\frac{8!}{5!(8-5)!}\times(243)=56\times(243)=13,608$

Example 3: Obtain and simplify the term in expansion of $(2x^2-y^3)^8$ which contains x^{10}

We are looking for the coefficient of x^{10} which is the 4th term $\Rightarrow r=4$

So, the 4th term is $\binom{8}{4-1}(2x^2)^{8-(4-1)}(-y^3)^{4-1}=\binom{8}{3}(2x^2)^{8-(4-1)}(-y^3)^{4-1}$

$$\Rightarrow \binom{8}{3}(2x^2)^{8-(4-1)}(-y^3)^{4-1}$$
$$=\frac{8!}{3!(8-3)!}(2x^2)^5(-y^3)^3$$
$$=(56)(32x^{10})(-y^9)$$
$$=-1792x^{10}y^9$$