Applying Matrices to Linear Systems

Matrix addition and matrix multiplication have many of the properties of ordinary addition and multiplication.

	Properties of real numbers	Properties of Matrices
Property	Let a, b, and c be real numbers	Let A, B, and C be $m\times n$ matrices
Commutative	$a+b=b+a$ $ab=ba$	$A+B=B+A$ $AB\neq BA$
Associative	$(a+b)+c=a+(b+c)$ $(ab)c=a(bc)$	$(A+B)+C=A+(B+C)$ $(AB)C=A(BC)$
Distributive	$a(b+c)=ab+ac=ba+bc$ $=(b+c)a$	$A(B+C)=AB+AC$ $(B+C)A=BA+CA$

An important exception to the similarity of these properties is that matrix multiplication is not commutative: in general $AB\neq BA$.Since products cannot commute, left multiplication can give a different product from right multiplication.
Before discussing other properties, we first need to identify some important matrices.

Any $m\times n$ matrix whose elements are all zero is called a **zero matrix**, denoted by $O_{m\times n}$. The following matrices are zero matrices.

$$O_{1\times3}=\begin{bmatrix}0&0&0\end{bmatrix}\qquad O_{2\times1}=\begin{bmatrix}0\\0\end{bmatrix}\qquad O_{2\times3}=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$$

A **square matrix** is any matrix having the same number of column and rows.
The main diagonal of a square matrix is the diagonal that extends from upper left to lower right.
Any $n\times n$ matrix whose main diagonal elements are 1 and whose other elements are 0 is called an identity matrix, denoted by $I_{n\times n}$. The following matrices are identity matrices:

$$I_{2\times2}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\qquad I_{3\times3}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Definition 1: **The Identity Matrix**, written *I*, is a square matrix where all the elements are 0 except the *principal diagonal* which has all ones
The identity matrix *I* is analogous to the number "1" in ordinary number multiplication. If we multiply the number 8 by 1 (on either side), we have no change - the answer is 8.
1 × 8 = 8 × 1 = 8

Definition 2: A **diagonal matrix** is a square matrix that has zeroes everywhere except along the main diagonal (top left to bottom right).
For example, here is a 3 × 3 diagonal matrix:

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Note: The **identity matrix** (above) is an example of a **diagonal matrix.**

Definition 3: Powers of square matrices are defined just like powers of real numbers:

$$A^n = \underbrace{A \bullet A \bullet A \bullet A \bullet A \bullet A \ldots . \bullet A}_{n \text{ factors}}$$

As with real numbers, matrix addition and multiplication both have identity properties. Matrix addition and multiplication have both inverse properties. The real numbers 2 and -2 are called additive inverses since 2+(-2)=0. Similarly, a matrix can have an additive inverse.

Definition 4: The additive inverse of a matrix A, denoted by –A, is the matrix in which each element is the opposite of its corresponding element in A.

Example 1: Find the additive inverse of the matrix A

$$A = \begin{bmatrix} 2 & 0 & -3 \\ -3 & 6 & 1 \end{bmatrix}, then - A = \begin{bmatrix} -2 & 0 & 3 \\ 3 & -6 & -1 \end{bmatrix}$$

The real numbers 2 and 2^{-1} are called multiplicative inverses since $2 \bullet 2^{-1} = 1$. Similarly, a matrix can have a multiplicative inverse. The examples below illustrate the identity and inverse properties for multiplication.

$$\begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding the multiplication inverse of a square matrix can involve a lot of computations.

Rule 2: The multiplication inverse of a square matrix A is A^{-1}

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1}=\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $|A|=ad-bc$.

$|A|$ is the determinant of A and $|A|\neq 0$

Check the table that summarize the properties of real numbers and matrices

	Properties of Real Numbers	Properties of Matrices
Properties of Multiplication	Let a be any real number	Let A be any $n\times n$ matrix. Let I be $n\times n$ the identity matrix and O be the $n\times n$ zero matrix
Identity	$a\bullet 1=1\bullet a=a$	$A\bullet I=I\bullet A=A$
Inverse	$a\bullet a^{-1}=a^{-1}\bullet a=1$ $a\neq 0$	$A\bullet A^{-1}=A^{-1}\bullet A=I$ $(\vert A\vert\neq 0)$
Multiplicative Property Of Zero	$a\bullet 0=0\bullet a=0$	$A\bullet O=O\bullet A=O$

Example 2: Find the multiplication inverse A^{-1} of a square matrix A. Check your answer by finding $A\bullet A^{-1}$

$$A=\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$$

$|A|=5\bullet 3-4\bullet 2=7. Since |A|\neq 0\Rightarrow A^{-1} exists$

$$A^{-1}=\frac{1}{7}\begin{bmatrix} 3 & -4 \\ -2 & 5 \end{bmatrix}=\begin{bmatrix} \frac{3}{7} & \frac{-4}{7} \\ \frac{-2}{7} & \frac{5}{7} \end{bmatrix}$$

$$A\bullet A^{-1}=\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}\bullet\begin{bmatrix} \frac{3}{7} & \frac{-4}{7} \\ \frac{-2}{7} & \frac{5}{7} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

Example 3: If $C=\begin{pmatrix}4 & 3\\ 2 & 3\end{pmatrix}$ find C^{-1}

$$ad-bc=12-6$$

$$=6$$

$$\therefore C^{-1}=\frac{1}{6}\begin{pmatrix}3 & -3\\ -2 & 4\end{pmatrix}$$

Solving matrix equations of the form: AX = B

You need to remember, that when you multiply A by its inverse, you get the identity matrix. So

$$A^{-1}\,A\,X = A^{-1}\,B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

The unknown matrix X, in the case above is referred to as a **RIGHT– HAND** factor, so to get X by itself, we multiply both sides on the **LEFT** by A^{-1}

Solving matrix equations of the form: XA = B

$$X\,A\,A^{-1} = B\,A^{-1}$$
$$X\,I = B\,A^{-1}$$
$$X = BA^{-1}$$

The unknown matrix X, in the case above is referred to as a **LEFT– HAND** factor, so to get X by itself, we multiply both sides on the **RIGHT** by A^{-1}

Example 4: Solve for x:

1) $X\begin{bmatrix}3 & 4\\4 & 5\end{bmatrix}=\begin{bmatrix}4 & -2\\1 & 6\end{bmatrix}$

$$X=\begin{bmatrix}4 & -2\\1 & 6\end{bmatrix}\begin{bmatrix}3 & 4\\4 & 5\end{bmatrix}^{-1} \quad \textit{note the placement of the inverse matrix}$$

$$X=\begin{bmatrix}4 & -2\\1 & 6\end{bmatrix}\frac{1}{-1}\begin{bmatrix}5 & -4\\-4 & 3\end{bmatrix}$$

$$X=\begin{bmatrix}-28 & 22\\19 & -14\end{bmatrix}$$

2) $\begin{bmatrix}3 & 5\\2 & 4\end{bmatrix}X=\begin{bmatrix}10 & 15\\12 & 24\end{bmatrix}$

$$X=\begin{bmatrix}3 & 5\\2 & 4\end{bmatrix}^{-1}\begin{bmatrix}10 & 15\\12 & 24\end{bmatrix} \quad \textit{note the inverse is on the LHS this time}$$

$$X=\frac{1}{2}\begin{bmatrix}4 & -5\\-2 & 3\end{bmatrix}\begin{bmatrix}10 & 15\\12 & 24\end{bmatrix}$$

$$X=\begin{bmatrix}-10 & -30\\8 & 21\end{bmatrix}$$