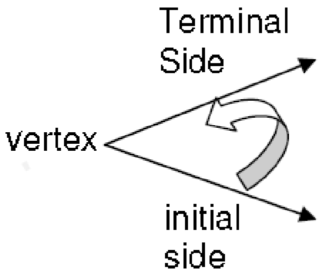
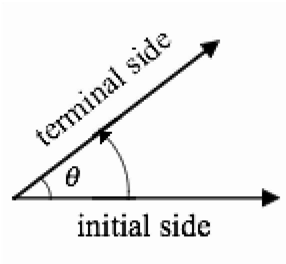
Radian and Degree Measure

As derived from the Greek language, the word Trigonometry means "measurement of triangles". Initially, trigonometry dealt with relationships among the sides and angles of triangles and was used in the development of astronomy, navigation, and surveying. Nowadays the application of trigonometry expanded to include a vast number of physical phenomena involving rotations and vibrations (sound waves, light rays, planetary orbits, vibrating strings, pendulums…)

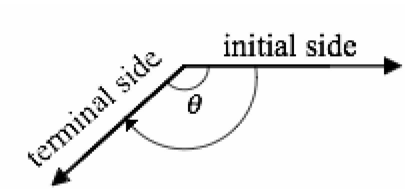
An **angle** is determined by rotating a ray about its endpoint. The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side**. The endpoint of the ray is the **vertex** of the angle.



This perception of an angle fits a coordinate system in which the origin is the vertex and initial side coincides with the positive x -axis. Such an angle is in **standard position**. **Positive angles** are generated by **counterclockwise** rotation, and **negative angles** by **clockwise** rotation.



positive angle



negative angle

The measure of an angle is determined by the amount of rotation from the initial side to the terminal side.

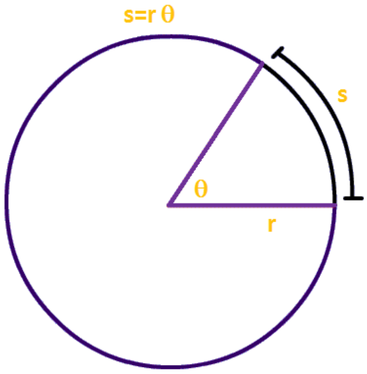
<u>Units of measuring angle</u>

There are two types of units:

1) Radian measure
2) Degree measure

One way to measure angles is in radians. This type of measure is especially useful in calculus. To define a radian, you can use a **central angle** of a circle, one whose vertex is the center of the circle.

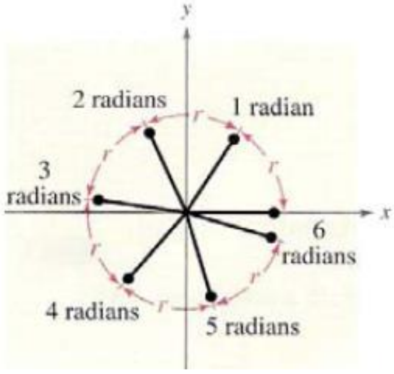
Definition 1: One radian is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle.



$$\theta = \frac{s}{r}$$

where θ is measured in radians

Because the circumference of a circle is $2\pi r$ units, it follows that a central angle of one full revolution (counterclockwise) corresponds to an arc length of s = $2\pi r$. Moreover, because $2\pi \approx$ 6.28, there are just over six radius lengths in a full circle.

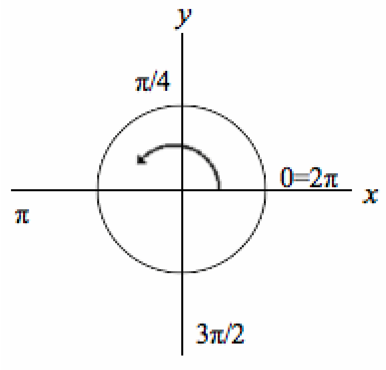


The radian measure of an angle of one full revolution is 2π, you can obtain the following:

$$\frac{1}{2}\ revolution = \frac{2\pi}{2} = \pi\ radians$$

$$\frac{1}{4}\ revolution = \frac{2\pi}{4} = \frac{\pi}{2}\ radians$$

$$\frac{1}{6}\ revolution = \frac{2\pi}{6} = \frac{\pi}{3}\ radians$$



Definition 2: Two angles are co terminal if they have the same initial and terminal sides.

For instance, the angles $\frac{\pi}{6}$ and $\frac{13\pi}{6}$ are co terminal. You can find an angle that is co terminal to a given angle θ by adding or subtracting 2π (one revolution). A given angle θ has infinitely many co terminal angles: $\theta + 2\pi n$ where n is an integer.

Example 1: Find a co terminal angle for each of the following:

1) $\frac{11\pi}{5}$

For the positive angle $\frac{11\pi}{5}$, subtract 2π : $\frac{11\pi}{5}-2\pi=\frac{11\pi-10\pi}{5}=\frac{\pi}{5}$

2) $\frac{2\pi}{7}$

For the positive angle $\frac{2\pi}{7}$, subtract 2π : $\frac{2\pi}{7}-2\pi=\frac{2\pi-14\pi}{7}=\frac{-12\pi}{7}$

3) $-\frac{2\pi}{3}$

For the negative angle $-\frac{2\pi}{3}$, add 2π : $-\frac{2\pi}{3}+2\pi=\frac{-2\pi+6\pi}{3}=\frac{4\pi}{3}$

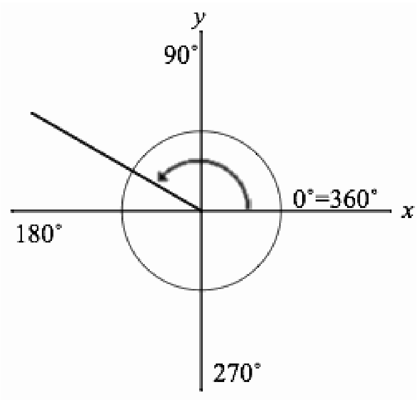
Definition 3: Two positive angles α and β are complementary if their sum is $\frac{\pi}{2}$. Two positive angles are supplementary if their sum is π .

Example 2: Find the complement and the supplement of $\frac{4\pi}{5}$ if possible:

$\frac{4\pi}{5}$ is greater than $\frac{\pi}{2} \Rightarrow$ It has no complement

The supplement of $\frac{4\pi}{5}$ is: $\pi-\frac{4\pi}{5}=\frac{5\pi-4\pi}{5}=\frac{\pi}{5}$

A second way to measure angles is in terms of degrees, denoted by the symbol ^{0}. A measure of one degree $\left(1^0\right)$ is equivalent to a rotation of $\frac{1}{360}$ of a complete revolution about the vertex. To measure angles, it is convenient to mark degrees on the circumference of a circle. A full revolution (counterclockwise) corresponds to 360^0, a half revolution to 180^0, a quarter revolution to 90^0, and so on.



To get a more exact measurement, fractional parts of degrees (^{0}) are expressed in minutes $(')$ and seconds $('')$

$$1' = one \text{ minute} = \frac{1}{60} \text{ of a degree}$$

$$1'' = one \text{ second} = \frac{1}{3600} \text{ of a degree}$$

Converting Minutes and Seconds to Decimal Form

To convert from seconds to a decimal part of a minute, divide the number of seconds by 60.

To convert from minutes to a decimal part of a degree, divide the number of minutes by 60.

Example 3: Write the angle measurement in degree form

1) $64^0 47'$

$$64^0 47' = 64^0 + (47 \div 60)^0 = 64^0 + 0.783^0 = 64.783^0$$

2) $32^0 12' 10''$

$$\begin{aligned}
&32^0 12' 10'' \\
&= 32^0 12' + 10'' \\
&= 32^0 12' + (10 \div 60)' \\
&= 32^0 12.167' \\
&= 32^0 + 12.167' \\
&= 32^0 + (12.167 \div 60)^0 \\
&= 32^0 + 0.203^0 \\
&= 32.203^0
\end{aligned}$$

Because 2π radians corresponds to one complete revolution, degrees and radians are related by the equations $360^0 = 2\pi$ rad. and $180^0 = \pi$ rad.

This leads to the conversion rule from radians to degrees and from degrees to radians

$$1^0 = \frac{\pi}{180} \text{ rad.} \qquad 1 \text{ rad.} = \frac{180^0}{\pi}$$

Rule 1: Conversions between Degrees and Radians

1) To convert **degrees** to **radians**, multiply degrees by $\frac{\pi \text{ rad.}}{180^0}$
2) To convert **radians** to **degrees**, multiply radians by $\frac{180^0}{\pi \text{ rad.}}$

Remark: If there is no unit to specify the angle's measurement, radian measure is implied. For instance, if you write $\theta=3$, you imply that $\theta=3$ radians

Example 4: Convert the angle measure from degrees to radians.

1) 135^0

$$135^0 = 135^0\left(\frac{\pi \text{ rad.}}{180^0}\right) = \frac{3\pi}{4} \text{ rad.}$$

2) 540^0

$$540^0 = 540^0\left(\frac{\pi \text{ rad.}}{180^0}\right) = 3\pi \text{ rad.}$$

3) -270^0

$$-270^0 = -270^0\left(\frac{\pi \text{ rad.}}{180^0}\right) = \frac{-3\pi}{2} \text{ rad.}$$

Example 5: Convert the angle measure from radians to degrees.

1) $-\frac{\pi}{3}$ rad.

$$-\frac{\pi}{3} \text{ rad.} = -\frac{\pi}{3} \text{ rad.}\left(\frac{180^0}{\pi \text{ rad.}}\right) = -60^0$$

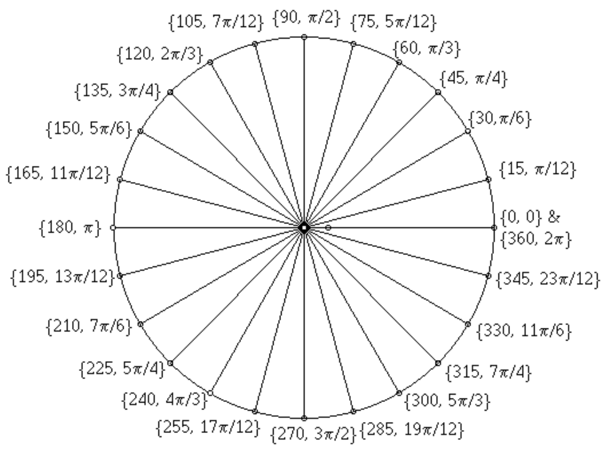
2) $\frac{5\pi}{6}$ rad.

$$\frac{5\pi}{6} \text{ rad.} = \frac{5\pi}{6} \text{ rad.}\left(\frac{180^0}{\pi \text{ rad.}}\right) = 150^0$$

3) $\frac{\pi}{9}$ rad.

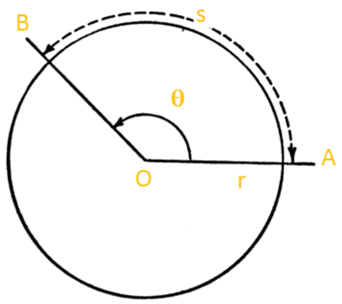
$$\frac{\pi}{9} \text{ rad.} = \frac{\pi}{9} \text{ rad.}\left(\frac{180^0}{\pi \text{ rad.}}\right) = 20^0$$

The figure below gives the relationship between degrees and radians for the most common angles in the unit circle measured in the counterclockwise direction from the point to the right of the vertex. The form of the ordered pair is {degree measure, radian measure}

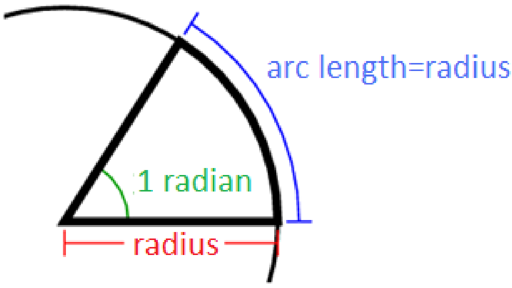


The radian measure formula $\theta = \frac{s}{r}$ can be used to measure arc length along a circle.

Definition 4: For a circle of radius r and center O, a central angle θ intercepts an arc of length s given by: $s = r\theta$; where θ is measured in radians.



Note: If r=1, then $s = \theta$, and the radian measure of θ equals the arc length.
If θ =1, then $r = s$, and the radian measure of θ equals the arc length.



Example 6: A circle has a radius of 4 cm. Find the length of the arc intercepted by a central angle of 240°

$$\theta = 240^\circ = (240\ \text{deg})\left(\frac{\pi\ rad}{180\ \text{deg}}\right) = \frac{4\pi}{3} rad.$$

$$s = r\theta = 4\left(\frac{4\pi}{3}\right) = \frac{16\pi}{3} \approx 16.76\ \text{cm}$$

Definition 5: For a circle of radius r and center O, the area A of the circle with central angle θ is given by: $A=\frac{1}{2}r^2\theta$ where θ is measured in radians.

Example 7: A sprinkler on a golf course fairway is set to spray water over a distance of 70 feet and rotates through an angle of 120°. Find the area of the fairway watered by the sprinkler.

$$\theta=120^\circ=(120\ \deg)\left(\frac{\pi\ rad}{180\ \deg}\right)=\frac{2\pi}{3}rad.$$

$$A=\frac{1}{2}r^2\theta=\frac{1}{2}(70)^2\left(\frac{2\pi}{3}\right)=\frac{4900\pi}{3}\approx 5131\,ft^2$$